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A MATHEMATICAL MODEL FOR CALCULATING
NON-DETECTION PROBABILITY OF A
RANDOM TOUR TARGET

by

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This model assumes that there is a stationary searcher having a "cookie-cutter" sensor located in the center of the search area.

A Monte-Carlo simulation program was used to generate the non-detection probabilities. The output of this program was used to construct the required mathematical model.

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A Mathematical Model for Calculating Non-Detection Probability of a Random Tour Target

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I. RANDOM TOUR MODEL

A. INTRODUCTION

The main objective of this thesis was to construct and test an experimental mathematical model to predict the probability that a target moving according to a two-dimensional random tour will avoid detection to time t by a fixed sensor.

B. DESCRIPTION OF RANDOM TOUR MODEL

1. The Searcher Location

The searcher is assumed to be located in the center of a square search region of area A. This location is held fixed during the search period. The searcher has a detection capability over a disk of radius R. (See Figure 1.1).

The detection probability for a target inside this disk is 1 and it is 0 outside. The searcher thus has a "cookie-cutter" sensor with detection range R. [Ref. 1]

2. The Target Starting Position

The target's starting position is uniformly distributed over the square search region A.

3. Motion of the Target

The target moves randomly over the area A according to a random tour which reflects off the area boundaries. [Ref. 1]

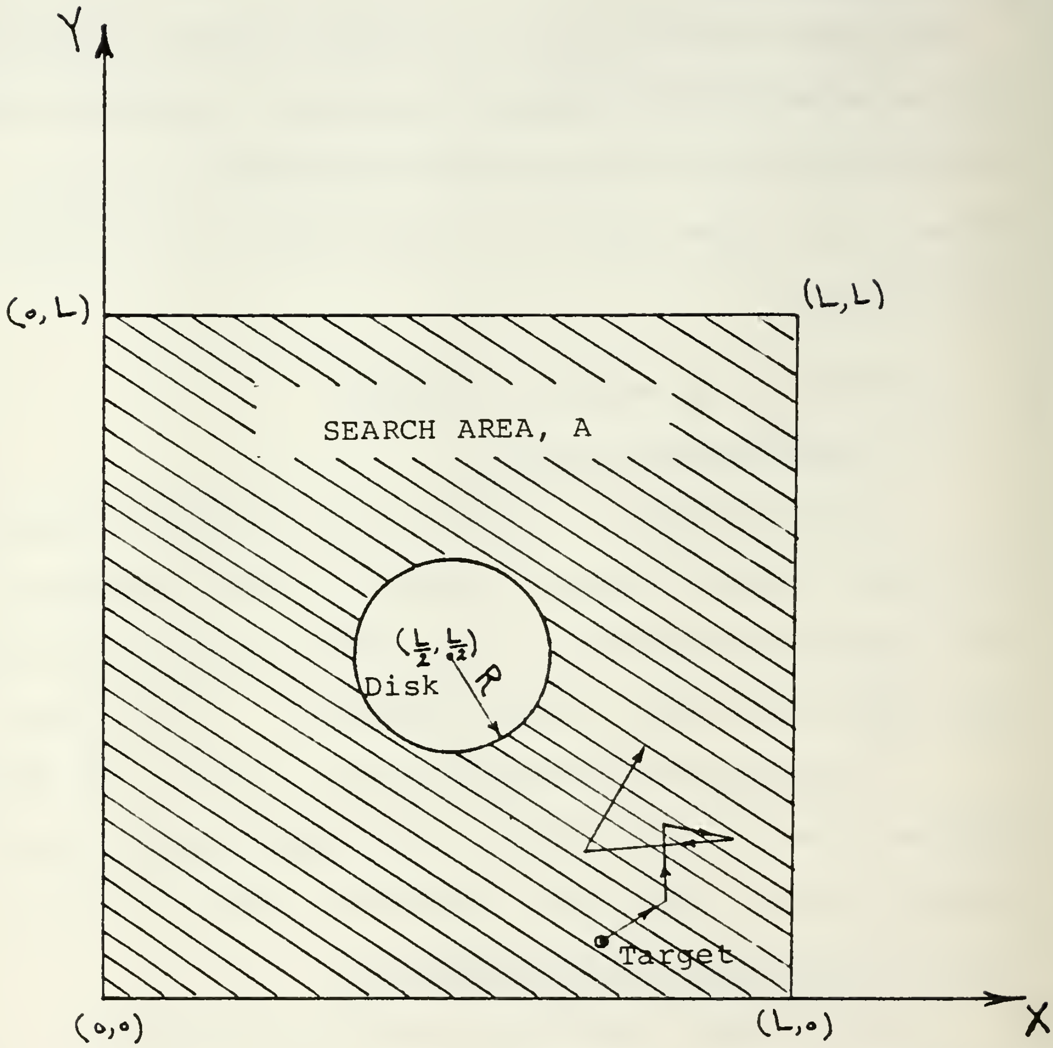


Figure 1.1 Random-Tour Model

The target track is a connected sequence of line segments. The direction, or target course θ, for each straight segment is selected from an independent uniform distribution between 0 and 2π radians.

The length of time, T, the target spends on each leg (assuming no reflection off the area boundaries) is selected from an independent exponential distribution with mean $1/\lambda$. The term λ is the rate of course change (again ignoring reflections).

4. <u>Detection</u>

Detection occurs the first time the target enters the searcher's detection disk; that is, when the distance between the target and the center of A is less than or equal to R.

C. NECESSITY OF SIMULATION

An analytic expression for the probability density of the target's position after a random tour of time length t was derived in [Ref. 2]. Given the target's initial position at the origin of a two dimensional coordinate system, this expression is:

$$g(x,y,t) = [e^{-\lambda t}/2\pi(Vt)^2]\{\delta(r-1) + [\lambda t/\sqrt{1-r^2}]\exp(\lambda t\sqrt{1-r^2})\} \quad . \tag{1.1}$$

where

V = Target speed (nautical miles per hour)

λ = Rate of course change (1/hour)

t = Time (hours)

$$r^2 = \frac{x^2 + y^2}{(Vt)^2}$$

δ = Dirac δ-function

x,y = Components of the target's new position.

Expression (1.1) does not account for boundary effects and it considers the initial position of the target to be the origin. Adding the effects of boundary reflection and assuming the initial starting position to be uniformly distributed over A significantly complicates the calculation of $g(x,y,t)$.

In addition, it was the purpose of this work to find the probability of non-detection to time t (PND(t)), not the probability density function for the target. Thus, it was necessary to use simulation to attack this problem.

D. SIMULATION MODEL OF RANDOM TOUR

A Monte-Carlo simulation computer model (called Random Tour Simulation or RATSIM) was used to estimate PND(t) for the random tour model. This program was written in FORTRAN and designed to run on the IBM 3033 at the Naval Postgraduate School. It uses the International Mathematical and Statistical Library (IMSL) packages GGUBS to generate uniform random variables and GGEXN to generate exponential random variables.

1. Inputs

- Radius of detection disk R, in nautical miles (nm).
- Area size A, in square nautical miles (nm^2).
- Target speed V, in nautical miles per hour (nm/hr).
- Rate of course change λ, in 1/hour (hr^{-1})
- Number of replications (REP).
- Detection period (TMAX), in hours (hr).
- Time increment ΔT, in minutes.

2. Functioning of the Program

(i) At the start of each replication, the initial starting position of the target is drawn from a uniform distribution over the area A. The course θ is drawn from a uniform distribution on $(0, 2\pi)$.

(ii) The course is changed after a random time leg T drawn from an exponential distribution with mean $1/\lambda$.

(iii) After each time increment Δt, the new position of the target is calculated from:

$$X_{new} = X_{old} + V \cdot \Delta t \cdot \sin\theta$$

$$Y_{new} = Y_{old} + V \cdot \Delta t \cdot \cos\theta$$

where

X_{new}, Y_{new} = coordinates of the new position at the end of Δt.

X_{old}, Y_{old} = coordinates of the old position at the beginning of Δt.

Also, the distance D between the new position of the target and the center of the searcher disk is calculated from:

$$D^2 = (X_{new} - X_{ser})^2 + (Y_{new} - Y_{ser})^2 \ .$$

where

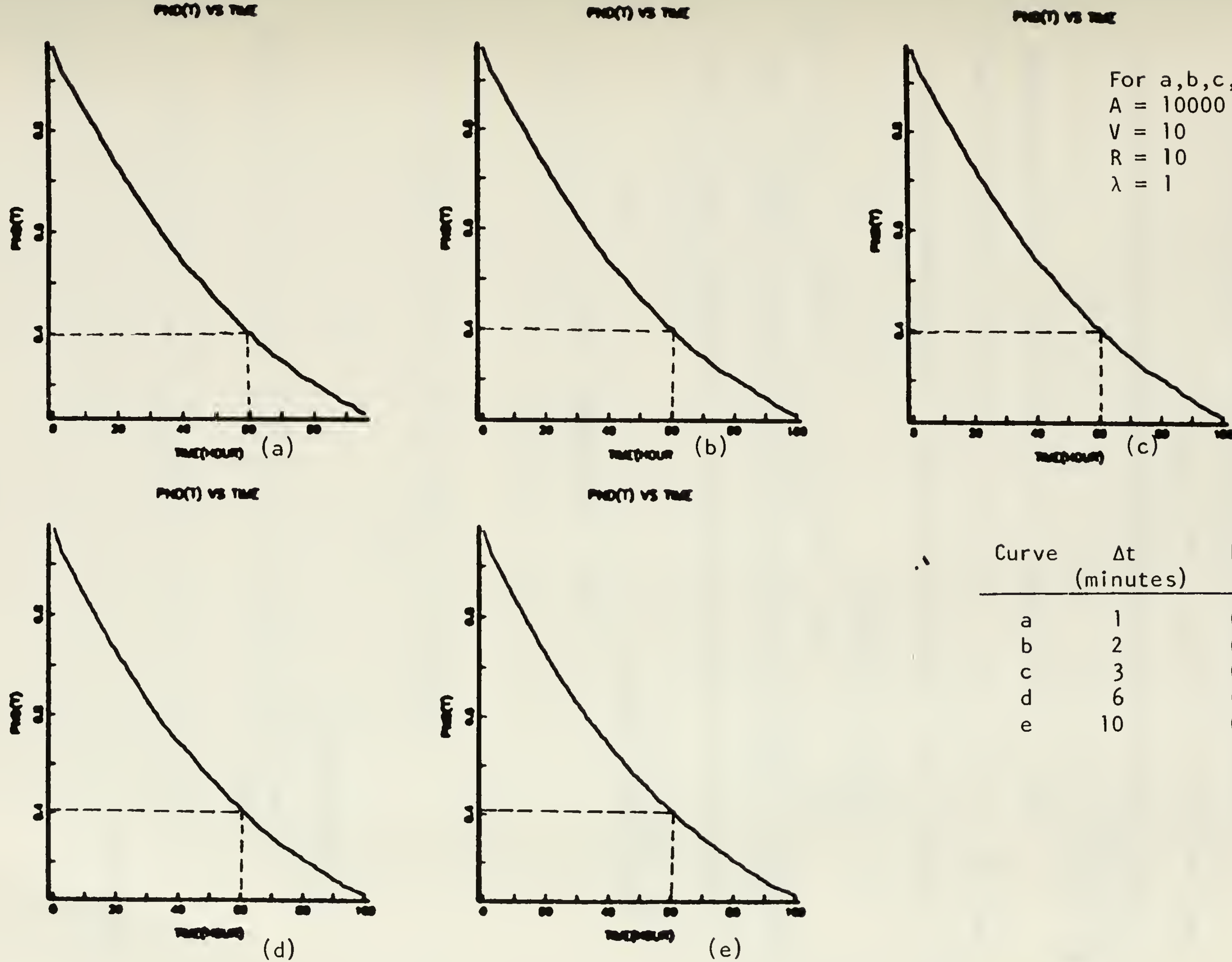
X_{ser}, Y_{ser} = coordinates of the center of the searcher's disk.

(iv) The replication terminates if: $D \leq R$ or if the detection period (TMAX) is over. Then a new replication begins. The process continues until the specified number of replications is reached.

(v) Two counters are used, one to determine the current time t, and the other to count the number of replications in which detection occurs.

3. Design of the Experiment

Different time increments Δt, varying from 1 minute up to 10 minutes, were tested with RATSIM and 3 minutes was accepted as a reasonable compromise. For smaller Δt, the execution time of the program increased unacceptably. For larger values, it was possible for the simulated path to jump across a significant portion of the detection disk without achieving detection, even though the line segment connecting two successive discrete positions of the target was partly on the disk [Ref. 3]. This will reduce the detection rate, especially for large V. However, as illustrated in Figure 1.2, PND(t) can be relatively insensitive to Δt less than 10 minutes when the problem



For a,b,c,d and e
$A = 10000 \quad nm^2$
$V = 10 \quad nm/hr$
$R = 10 \quad nm$
$\lambda = 1 \quad hr^{-1}$

Curve	Δt (minutes)	PND (60 hr)
a	1	0.3971
b	2	0.3974
c	3	0.398
d	6	0.412
e	10	0.418

Figure 1.2 PND(t) for Various Time Increments Δt

parameters are appropriate for antisubmarine warfare (ASW) search.

It was decided to conduct 2400 replications for each RATSIM experiment. This resulted in the standard deviation of the simulated PND(t) being no greater than $\sqrt{0.25/2400} \approx 0.0102$.

Also the maximum time allowed for detection (TMAX) was set at 100 hours. This was selected to allow PND(TMAX) to be near 0 for all tested values of problem parameters.

4. Boundary Effects

When the target encounters a boundary, a reflection is made to keep the target inside the search area A. The target position after reflection is determined as follows:

In Y-Direction:

If $Y < 0$ then Y becomes $(-Y)$,

If $Y > L$ then Y becomes $(2L - Y)$.

where

L = length of a side of the square search area A; i.e., $L = \sqrt{A}$.

In X-Direction:

The target reflects in the X-direction in a similar manner.

The target course θ changes after reflection as follows:

At $Y = 0$ or $Y = L$: θ bcomes $(2\pi - \theta)$,

At $X = 0$ or $X = L$: θ becomes $(\pi - \theta)$.

Thus, "the angle of incidence equals the angle of reflection." The reflection process is illustrated in Figure 1.3.

5. Output

At each time t, the primary simulation output is the ratio $\frac{N_T - N_0}{N_T}$ where

N_0 = number of replications giving a detection by time t, and

N_T = total number of replications used in Monte-Carlo simulation.

This ratio is the simulated probability of non-detection by time t, PND(t).

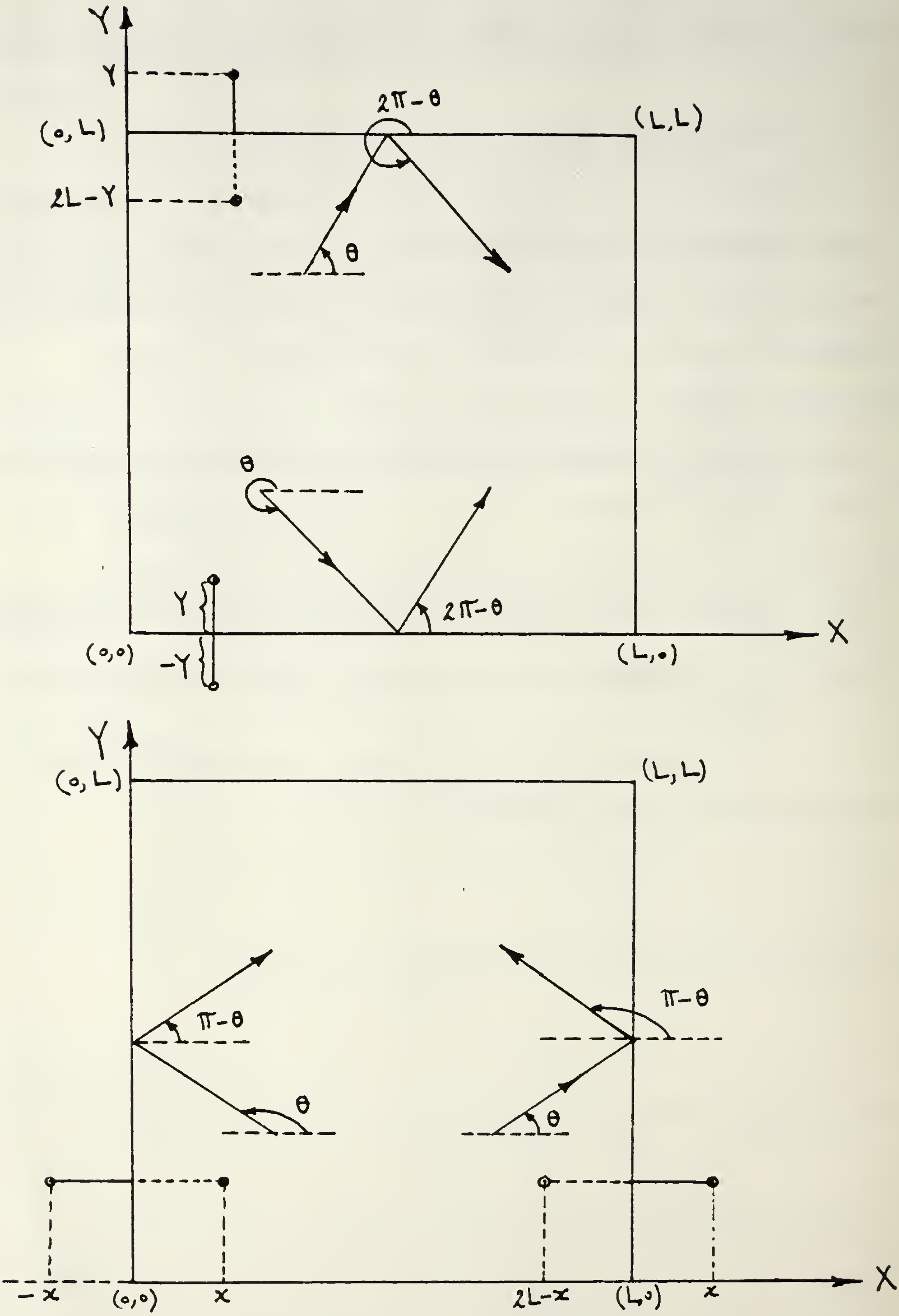


Figure 1.3 Reflection Process

II. RELATIONSHIP BETWEEN THE RANDOM TOUR AND DIFFUSION MODELS

A. DESCRIPTION OF DIFFUSION MODEL

In the diffusion model considered here, the target moves randomly over a square search area A according to Brownian motion with a diffusion constant D (units: area/time). Perfect reflection occurs at the area boundaries.

The target starting position is uniformly distributed over A. For any time interval of length Δt which does not contain a boundary reflection, the components of the target's position along the X and Y axes suffer increments which are each distributed independently and normally with mean 0 and variance $D\ \Delta t$.

A searcher having a "cookie-cutter" sensor with detection range R is located at the center of the search region.

Detection occurs whenever the range between the searcher and the target becomes R or less.

B. RELATIONSHIP WITH RANDOM TOUR MODEL

In [Ref. 4] it is shown that as the rate of course change λ for an unconstrained random tour gets larger such that V^2/λ = constant, then the random tour can be approximated by a diffusion model with a diffusion constant

$D = V^2/\lambda$. In this case, the two models are said to be "equivalent".

Also, it is argued in [Ref. 5] that the detection probability predicted by a constrained (by reflecting boundaries) diffusion model represents an upper bound to that predicted by the equivalent constrained random tour model. In other words, the non-detection probability predicted by a constrained diffusion model is a lower bound to that predicted by the equivalent constrained random tour model. This is reasonable since it is known [Ref. 4] that the target in an unconstrained diffusion model will "on average" move a greater distance from the origin than a target conducting the equivalent random tour. Consequently, the diffusing target would be expected to encounter a stationary searcher more quickly.

These observations, as regarding the relationship between random tour and its equivalent diffusion, were supported by plotting the results of two simulation programs: RATSIM and DIFSIM.

DIFSIM (diffusion simulation) is a Monte-Carlo search simulation for a diffusing particle developed by Sislioglu [Ref. 5].

To generate the results displayed in Figure 2.1, the parameters A and R were held fixed at 1000 nm^2 and 28.2 nm respectively for both programs. The diffusion constant D

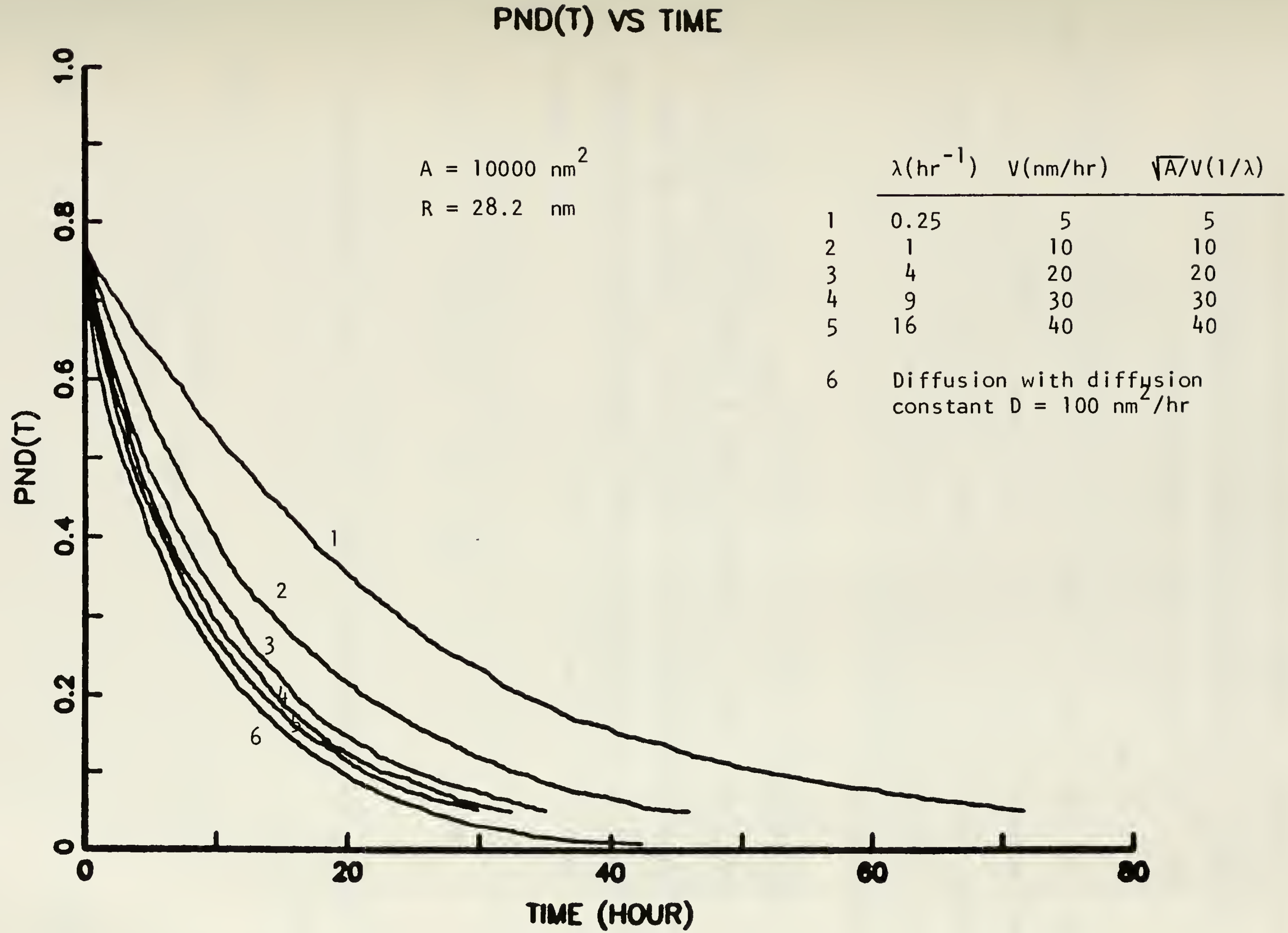


Figure 2.1 Asymptotic Approach of Random Tour to Diffusion

used with DIFSIM was 100 nm^2/hr. In RATSIM five different (λ, V) pairs were selected such that $V^2/\lambda = 100\ nm^2/hr$ for each pair. The values of $(\lambda\ hr^{-1}, V\ nm/hr)$ were (0.25, 5), (1, 10), (4, 20), (9, 30), and (16, 40).

It is clear from Figure 2.1 that, in this case as the ratio of the characteristic length of the search area to the mean segment length of the random tour ($\sqrt{A}/V(1/\lambda)$ gets larger the non-detection probability curves for a random tour model asymptotically approach that of the equivalent diffusion model.

C. MATHEMATICAL MODEL OF DIFFUSION

Sislioglu [Ref. 5] established a mathematical model to predict the probability of detection of a target moving according to the diffusion model (described in section A). This mathematical model is given by:

$$PD(t) = 1 - (1 - \frac{\pi R^2}{A}) \exp [- \frac{24.7\ RDt}{A^{1.5}}] \quad .$$

It was later modified by Eagle [Ref. 3] to the following form:

$$PD(t) = 1 - (1 - \frac{\pi R^2}{A}) \exp [\frac{-24.7\ RDt}{(A - \pi R^2)^{1.5}}] \quad .$$

where:

$PD(t)$ = probability of detection at time t in hr.

D = diffusion constant, nm^2/hr

R = radius of searcher disk, nm

A = area of search region, nm^2

So PND(t) can be given by:

$$PND(t) = 1 - PD(t)$$

$$PND(t) = \left(1 - \frac{\pi R^2}{A}\right) \exp\left[\frac{-24.7\ RDt}{(A - \pi R^2)^{1.5}}\right] . \tag{2.1}$$

As stated before, PND(t) as given by (2.1) should represent a lower bound on PND(t) as predicted by the equivalent random tour model, and will be used later in the next chapter as a basis to derive the mathematical model of random tour.

For simplicity, equation (2.1) will be written in the form:

$$PND(t) = \alpha \cdot e^{-\beta t} . \tag{2.2}$$

where

$$\alpha = 1 - \frac{\pi R^2}{A} . \tag{2.3}$$

and

$$\beta = \frac{24.7\ RD}{(A - \pi R^2)^{1.5}} \quad . \tag{2.4}$$

As indicated in [Ref. 5], the diffusion constant D can be approximated by V^2/λ to get a diffusion model equivalent to the random tour with V and λ. So, if we replace D by V^2/λ in (2.4) we get the approximate rate of detection for the equivalent diffusion model in the form

$$\beta = \frac{24.7\ RV^2}{(A - \pi R^2)^{1.5}\lambda} \quad . \tag{2.5}$$

III. THE ANALYTICAL MODEL FOR THE PROBABILITY OF NON-DETECTION

In this chapter an experimental analytical model is constructed to predict the probability of non-detection by time t of a target moving according to the random tour model described in Chapter I.

Simulation results from RATSIM will be used as well as the relationship between the random tour model and the asymptotically equivalent diffusion.

A. MODEL ASSUMPTIONS

The following assumptions are made:

1) The target starting position is uniformly distributed over the square search area A.

2) The target reflects perfectly off the area boundaries.

3) The target moves over the area A according to a random tour with constant speed V and rate of course change λ.

4) The searcher is fixed at the center of A.

5) The searcher detects with probability 1 all targets with a range of R or less. The searcher never detects targets at ranges greater than R. (That is, the searcher has a "cookie-cutter" sensor with detection range R [Ref. 2].)

6) The problem ends when the target is detected.

B. CLASSIFICATION OF VARIABLES

1. The Independent Variables

- Search area A in square nautical miles (nm^2).

- Target speed V in nautical miles per hour (nm/hr).
- Rate of course change λ in 1/hour (hr^{-1}).
- Searcher detection disk radius R in nautical miles (nm).

2. The Dependent Variables

- Probability of non-detection by time t, PND(t), i.e., PND(t) = f(A, V, R, λ).

C. CONSTRUCTION OF THE MODEL

By plotting PND(t) versus t, as estimated by RATSIM and with a logarithmic scale for the Y-axis, it was observed that the resulting curves were very nearly linear with negative slopes (see Figure 3.1).

This linear relationship on a logarithmic scale graph suggests the following functional form for PND(t):

$$\mathrm{PND}(t) = \alpha \cdot e^{-\gamma t} \quad . \qquad (3.1)$$

In the course of this research approximately 300 simulation experiments with RATSIM were conducted. All showed PND(t) to be approximately given by (3.1). Figures 3.2 through 3.5 are representative.

This thesis attempts to fit the simulation data and establish values of α and γ as functions of the problem independent variables A, V, R and λ.

A small subroutine was added to the main program of RATSIM to compute a least-squares estimate of α and γ. The formulas used [Ref. 6] were:

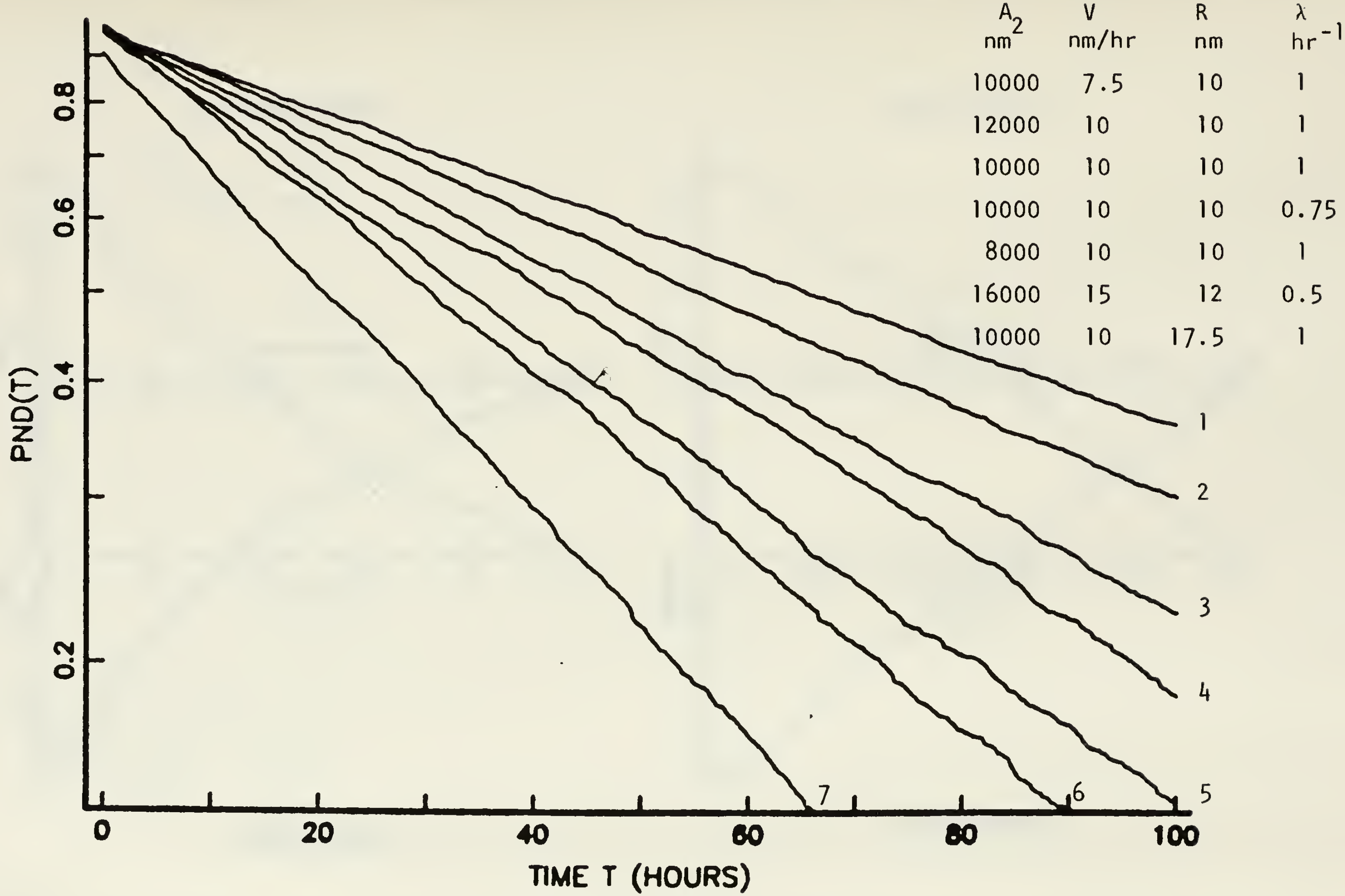
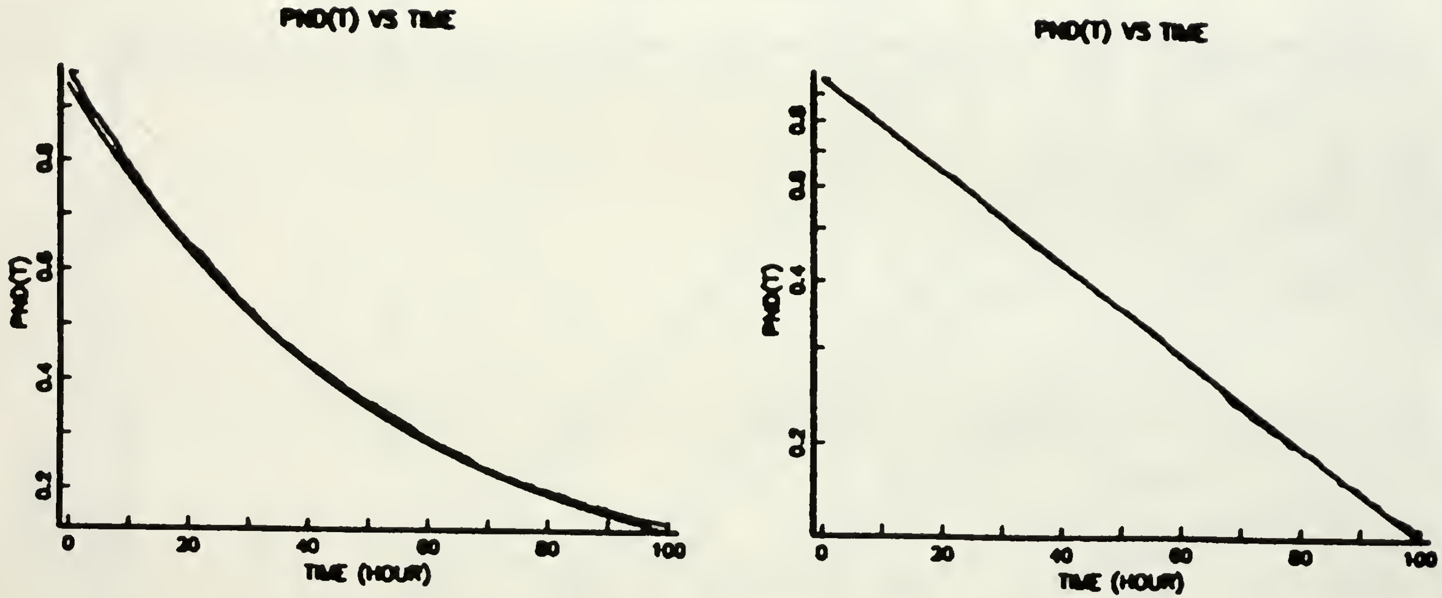


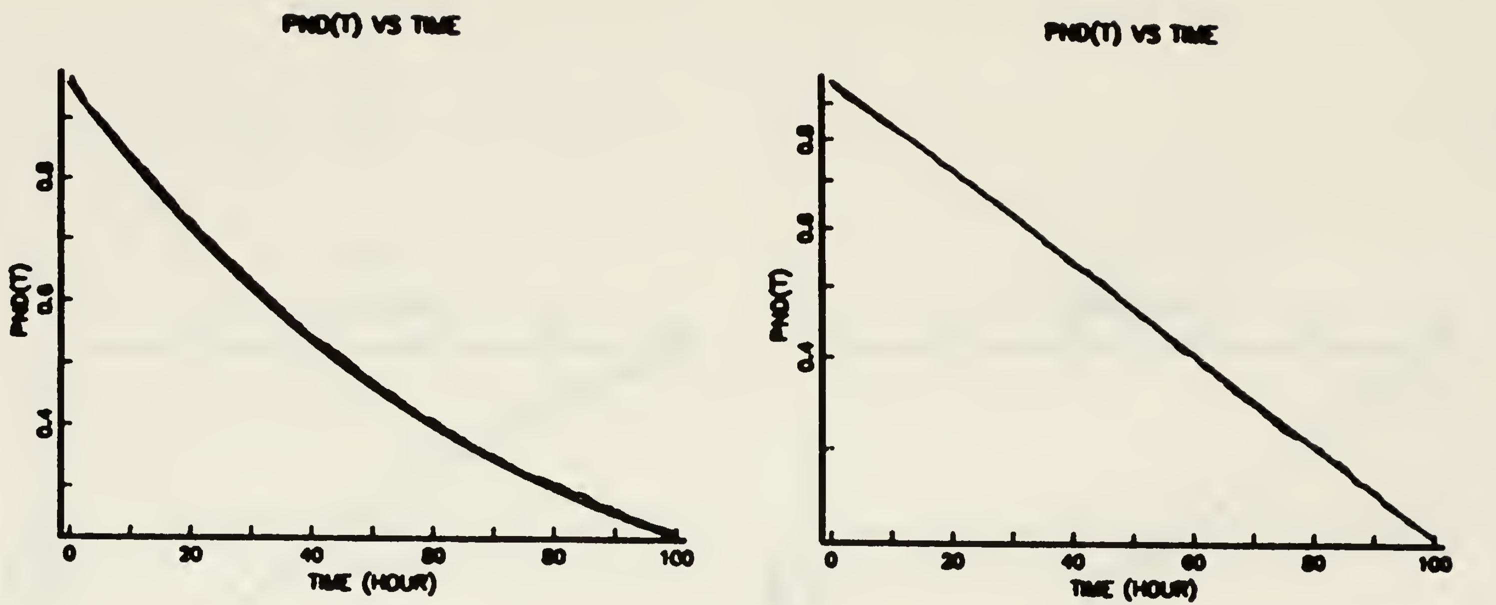
Figure 3.1 Probability of Non-detection by time t vs. t on a Log-scale for Y Axis



(a) (b)

$A = 6000 \quad nm^2$
$V = 8 \quad nm/hr$
$R = 8 \quad nm$
$\lambda = 0.5 \quad 1/hr$

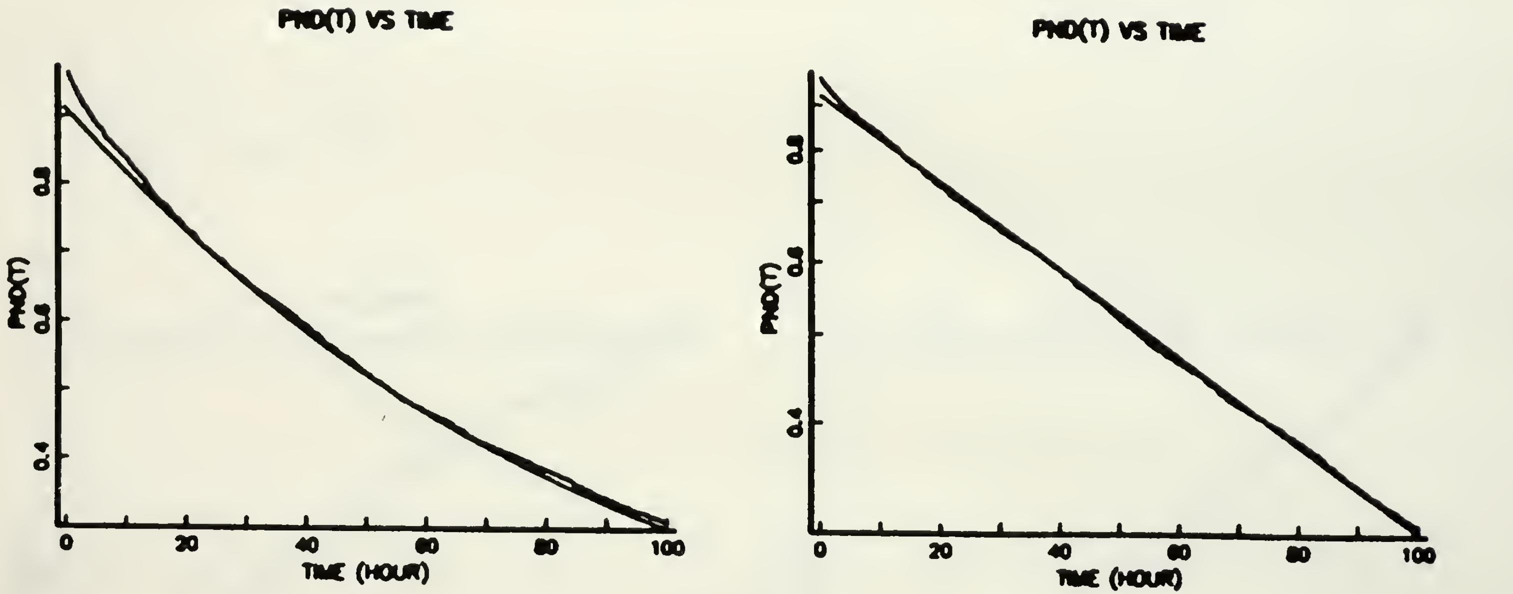
Figure 3.2 Exponential Fitting to Simulation Curve (PND(t) vs. t)
a) Lin-Scale b) Log-Scale



(a) (b)

$A = 10000$ nm^2
$V = 10$ nm/hr
$R = 10$ nm
$\lambda = 1$ 1/hr

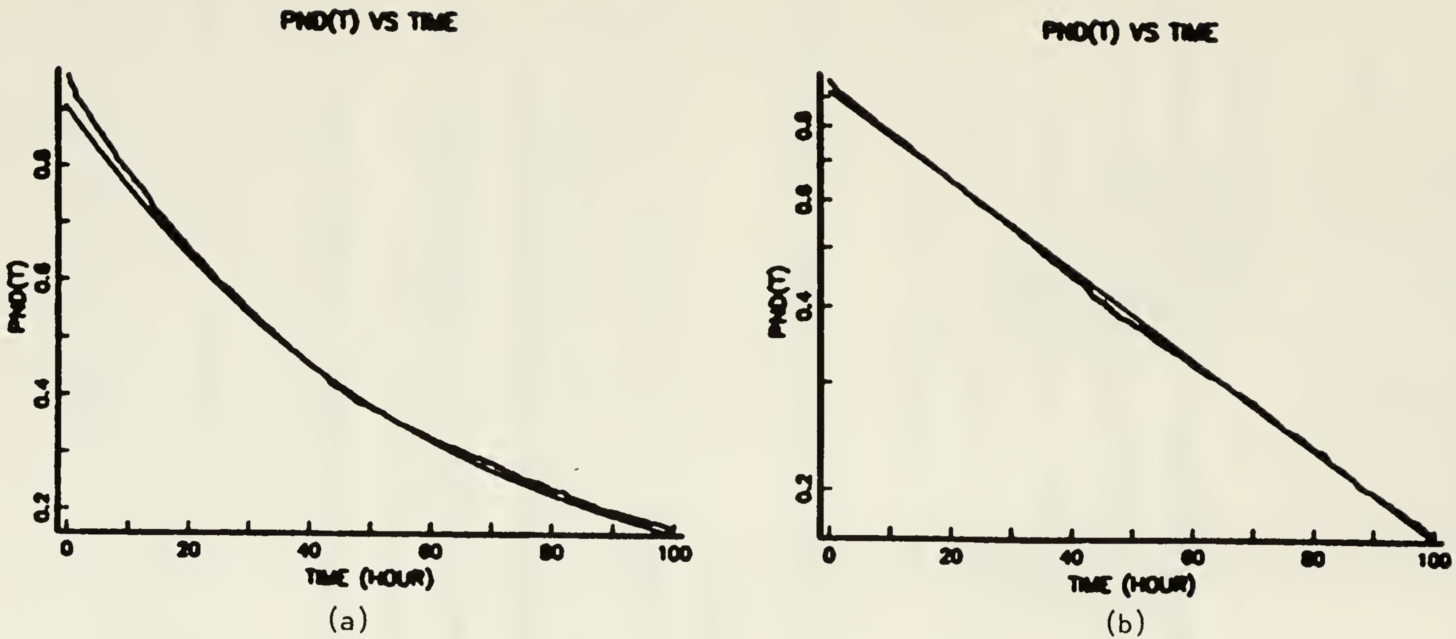
Figure 3.3 Exponential Fitting to Simulation Curve (PND(t) vs. t)
a) Lin-Scale b) Log-Scale



(a) (b)

$A = 14000$ nm^2
$V = 15$ nm/hr
$R = 12$ nm
$\lambda = 3$ 1/hr

Figure 3.4 Exponential Fitting to Simulation Curve (PND(t) vs. t)
a) Lin-Scale b) Log-Scale



$A = 20000 \quad nm^2$
$V = 20 \quad nm/hr$
$R = 15 \quad nm$
$\lambda = 2 \quad 1/hr$

Figure 3.5 Exponential Fitting to Simulation Curve (PND(t) vs. t)
a) Lin-Scale b) Log-Scale

$$\ln \alpha = \frac{\sum_{i=1}^{n} t_i \sum_{i=1}^{n} \ln PND(t_i) - \sum_{i=1}^{n} t_i \sum_{i=1}^{n} (t_i \ln PND(t_i))}{n \sum_{i=1}^{n} t_i^2 - (\sum_{i=1}^{n} t_i)^2} .$$

and

$$\gamma = \frac{n \sum_{i=1}^{n} (t_i \ln PND(t_i) - \sum_{i=1}^{n} t_i \sum_{i=1}^{n} \ln PND(t_i)}{n \sum_{i=1}^{n} t_i^2 - (\sum_{i=1}^{n} t_i)^2} .$$

where

n = number of data points used in the evaluation.

1. Submodel for α

Since the target starting position is uniformly distributed over the search area A, and the searcher has a perfect detection capability over a disk with area πR^2, we expect that immediately after the search begins the probability of detection will be $\pi R^2/A$.

If we substitute $t = 0$ in equation (3.1) we get

$$PND(0) = \alpha \cdot$$

So,

$$\alpha = 1 - \pi R^2/A \quad . \tag{3.2}$$

As would be expected, all simulations conducted showed

$$PND(0^+) \simeq 1 - \pi R^2/A \quad .$$

2. Submodel for γ

As stated before, PND(t) predicted by a diffusion model appears to be a lower bound to those values predicted by the equivalent random tour model.

A study of the simulation data suggests that γ in equation (3.1) can be estimated by:

$$\gamma = \beta(1 - e^{-\psi}) \quad . \tag{3.3}$$

where

- β is the detection rate of the equivalent diffusion model given by equation (2.5).
- ψ is a function of the independent problem variables A, V, R and λ.

In this thesis an attempt is made to find the functional relationship between ψ and the independent variables. It should be noted that there may be other functional forms for γ that fit the simulation data as well or better than equation (3.3).

3. Submodel for ψ

This submodel includes the A, R, V and λ. So it can be expected to be more complex than the submodel for α. To simplify the problem, the relationship between ψ and each one of these variables was investigated separately at first. Then a combination of these separate relationships was used to construct the required submodel for ψ.

a. Relationship Between ψ and Area Size A

To obtain this relationship, the variables V, R and λ were held fixed at 10 nm/hr, 10 nm and 1 hr^{-1} respectively. The area size A was varied between 900 nm^2 and 20000 nm^2.

Each simulation run required the independent variables A, R, V and λ to be specified, and gave a best fit for γ as an output. Then from equation (3.3) we have

$$\psi = - \ln \left(1 - \frac{\gamma}{\beta}\right) \cdot \tag{3.4}$$

Substituting (2.5) into (3.4) yields

$$\psi = -\ln \left\{1 - \frac{\gamma\lambda(A - \pi R^2)^{1.5}}{24.7 \, RV^2}\right\} . \tag{3.5}$$

By plotting the values of ψ, calculated by equation (3.5), versus the corresponding values of A, it was found that a power function fit the data very well (see

Figure 3.6). The least-squares best-fit ψ was found to be given by

$$\psi = 0.0080678(A)^{0.5028} \quad .$$

This implies that

$$\psi \propto \sqrt{A} \qquad (3.6)$$

where

"$\propto$" means "is proportional to".

b. Relationship Between ψ and Target Speed V

Here the variables A, R and λ were held fixed at 10000 nm^2, 10 nm, and 1 hr^{-1} respectively. V was then varied between 2.5 nm/hr and 25 nm/hr. The simulation output γ and equation (3.5) were used to generate the corresponding values of ψ.

By plotting ψ versus V and fitting a power function to the data, it was observed that

$$\psi = 8.3357(V)^{-1.001} \quad .$$

(See Figure 3.7). This implies that

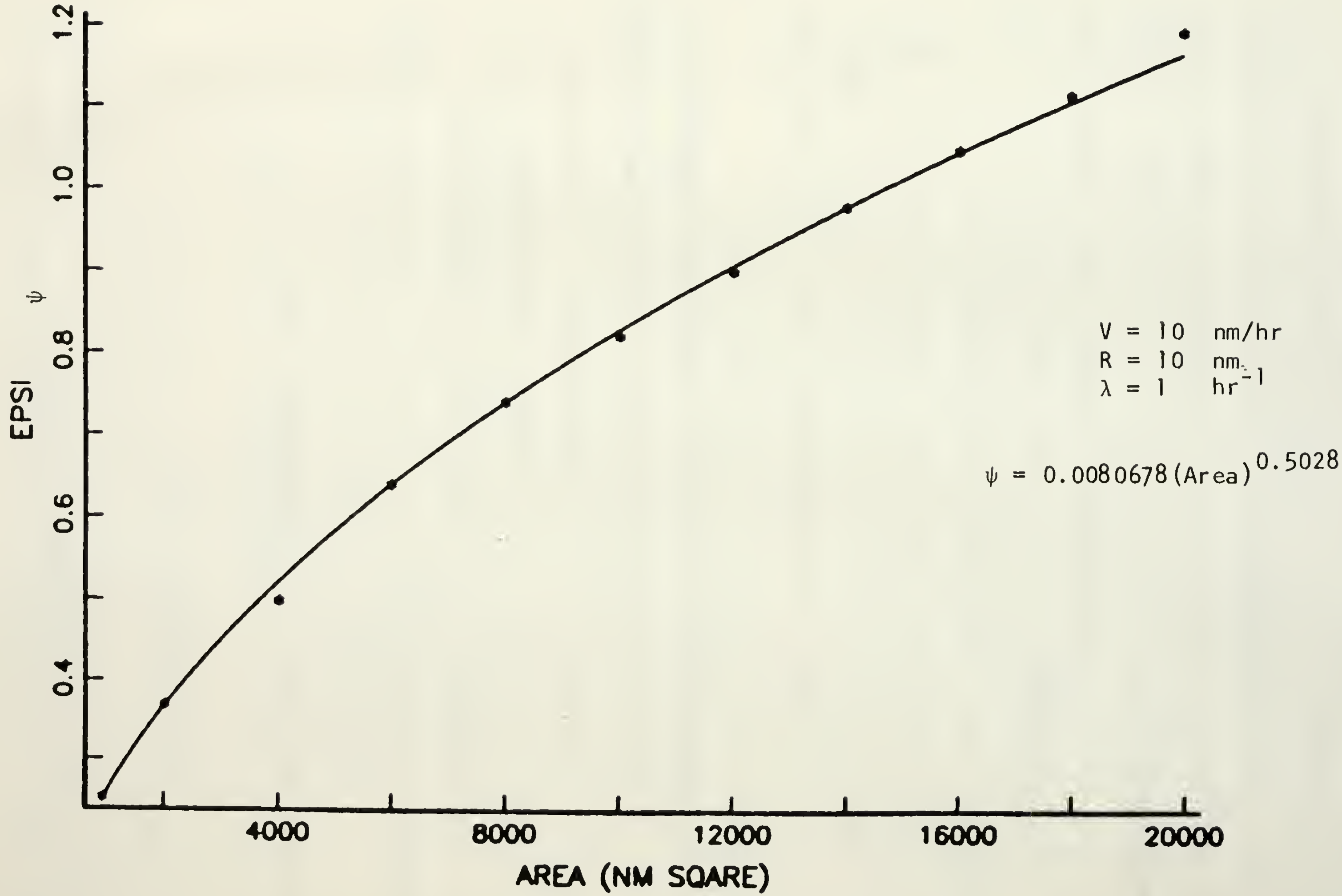


Figure 3.6 Variation of ψ with Area Size A

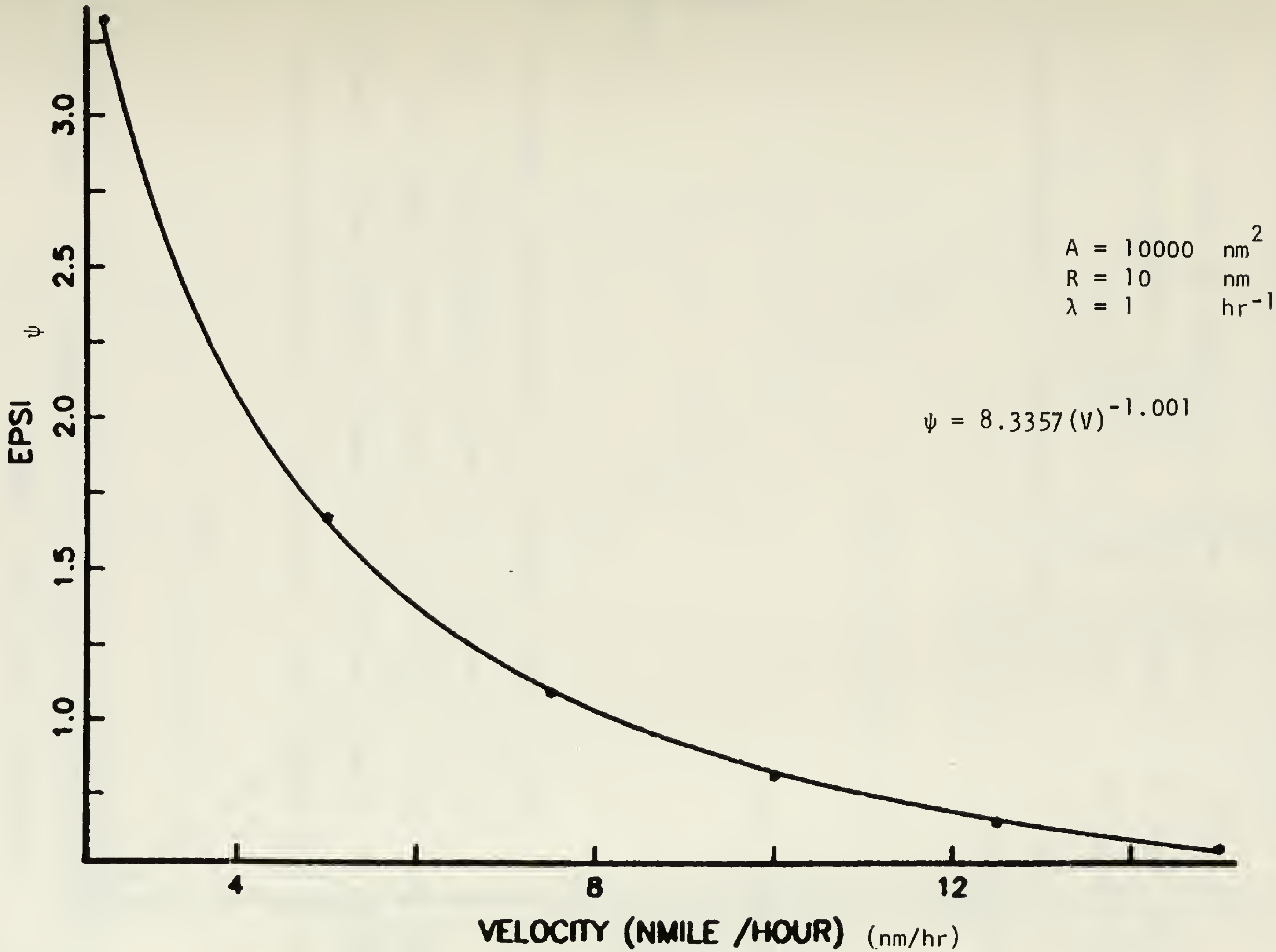


Figure 3.7 Variation of ψ with V

$$\psi \propto \frac{1}{V} \tag{3.7}$$

c. Relationship Between ψ and the Rate of Course Change λ

Now A, V and R were held fixed at 10000 nm^2, 10 nm/hr and 10 nm respectively. Then λ assumed the following values: 0.01, 0.05, 0.1, 0.2, 0.5, 1, 1.5, 2, 2.5, 3, and 4 hr^{-1}. The resulting best fit power function (see Figure 3.8) was found to be

$$\psi = 0.83419(\lambda)^{1.0003} \quad .$$

This means that

$$\psi \propto \lambda \tag{.38}$$

d. Relationship Between ψ and Detection Radius R

The following values were assigned to R: 2.5, 5, 7.5, 10, 12.5, 15, 20, 25, 30, and 35 nm. The other variables A, V and λ were held fixed at 10000 nm^2, 10 nm/hr and 1 hr^{-1} respectively. The best fit power function was

$$\psi = 0.8423(R)^{-0.009} \quad .$$

This indicates that ψ is nearly independent of R over this range of R values (see Figure 3.9).

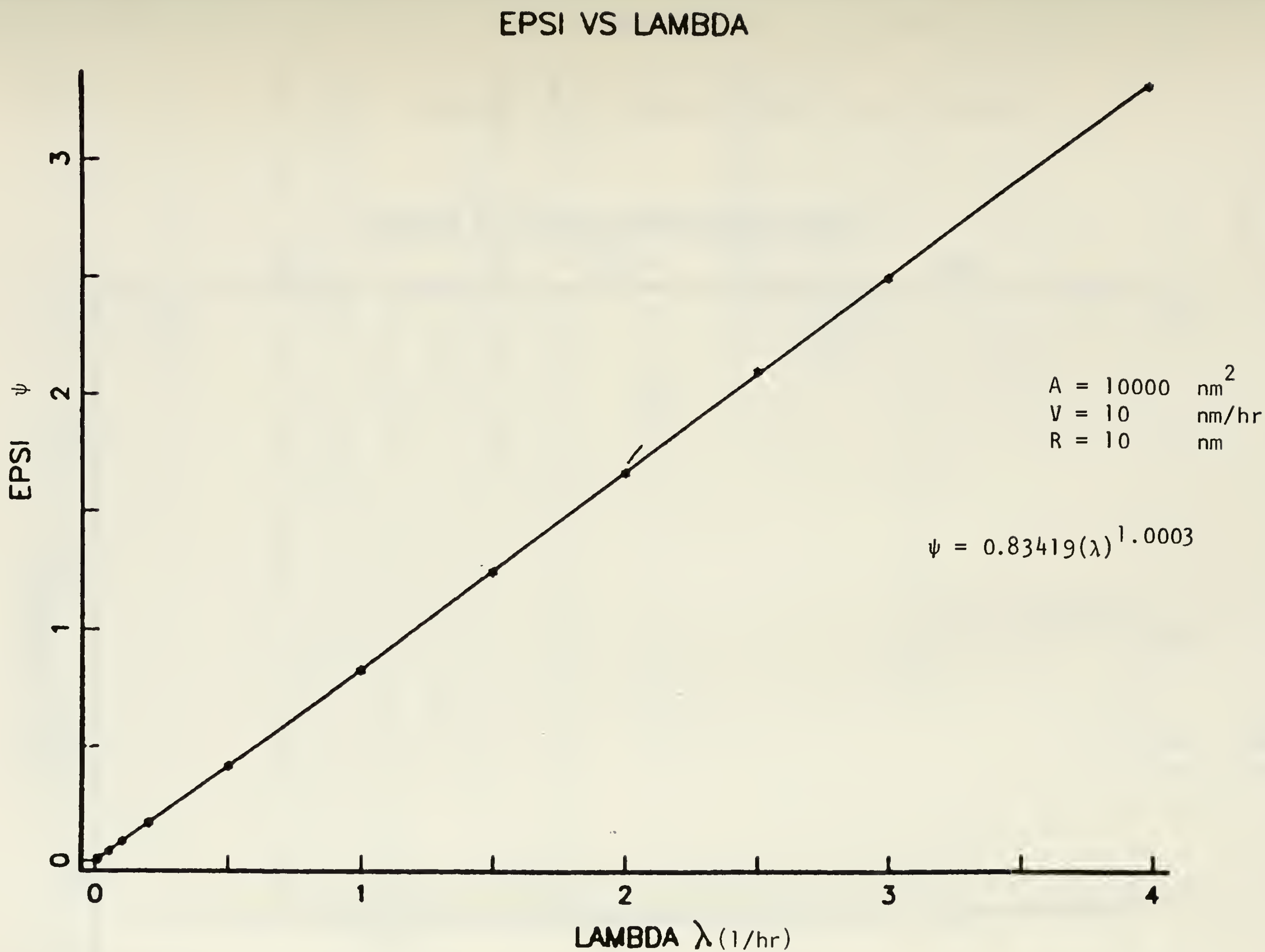


Figure 3.8 Variation of ψ with λ

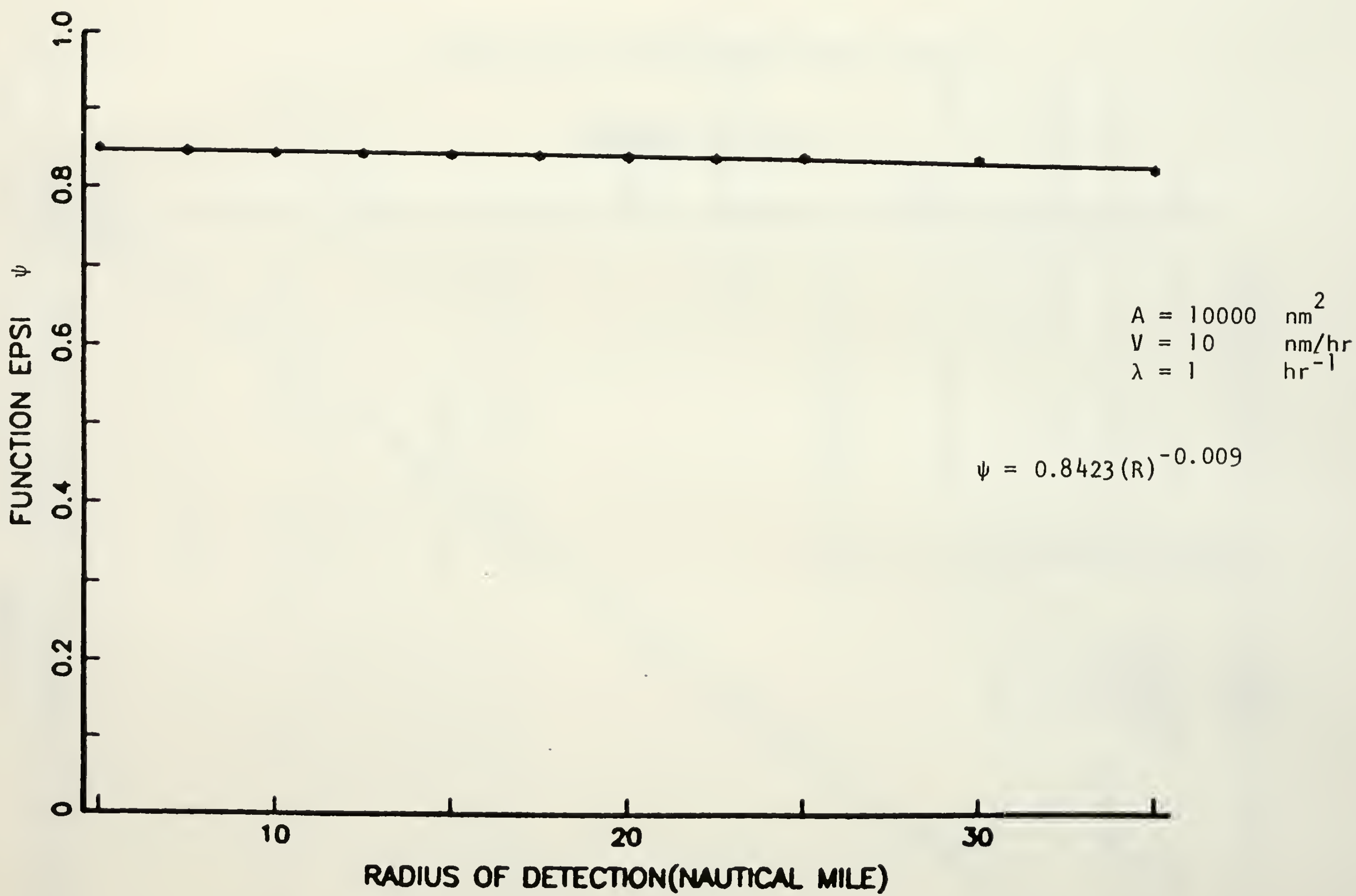


Figure 3.9 Variation of ψ with Radius R

e. Summary and Conclusions

Now, we can summarize the previous relationships as follows:

$$\psi \propto \sqrt{A} \ ,$$

$$\psi \propto 1/V \ ,$$

$$\psi \propto \lambda \quad .$$

This leads to the following conclusion:

$$\psi \propto \frac{\sqrt{A}\ \lambda}{V} \quad .$$

Or equivalently,

$$\psi = K \cdot \frac{\sqrt{A}\ \lambda}{V} \ . \tag{3.9}$$

where K is a proportionality constant to be estimated from the simulation data.

f. Estimation of the Coefficient K

The outputs γ of 156 simulation experiments with RATSIM were used to produce 146 sample K values. The value of K was calculated from equation (3.9) as follows:

$$K = \frac{\psi V}{\sqrt{A}\ \lambda} \ . \tag{3.10}$$

where the value of ψ was determined by equation (3.5).

The 156 RATSIM experiments used to estimate K resulted in a sample mean of 0.084 and a sample standard deviation of 0.0016. These data suggest that with probability 0.9, K lies in the interval [0.082, 0.087]. The bounds of this confidence interval were the observed 5 and 95 percentile points.

The histogram and the statistical summary table for this data are displayed in Figure 3.10.

g. Final Submodel for ψ

By estimating the value of K and applying equation (3.9) we can construct the final submodel for ψ as follows:

$$\psi = 0.084 \frac{\sqrt{A}\ \lambda}{V} \quad . \tag{3.11}$$

h. Final Submodel for γ

Substituting equations (3.11) and (2.5) into equation (3.3) we get

$$\gamma = \frac{24.7\ RV^2}{(A - \pi R^2)^{1.5}\lambda} [1 - \exp(-0.084 \frac{\sqrt{A}\ \lambda}{V})] \quad . \tag{3.12}$$

4. <u>The Final Form of the Random Tour Analytical Model</u>

Combining the final submodels for α and γ allows us to complete the random tour analytical model.

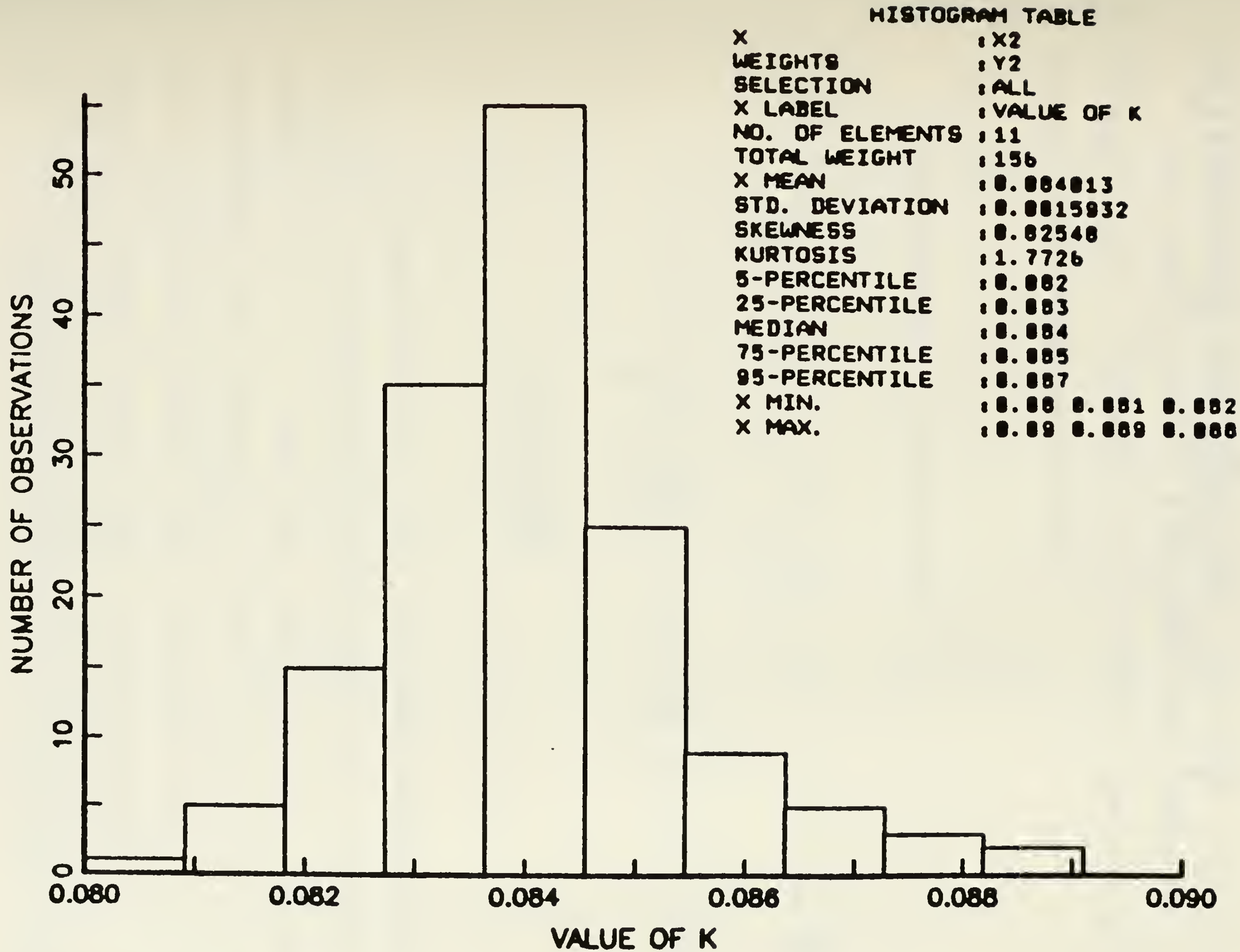


Figure 3.10 Distribution of Proportionality Constant K

Substituting equations (3.2) and (3.12) into (3.1) we get

$$PND(t) = \left(1 - \frac{\pi R^2}{A}\right) \mathrm{Exp}\left\{\frac{-24.7\ RV^2 t}{(A - \pi R^2)^{1.5}\lambda}\left[1 - \exp\left(-0.084\ \frac{\sqrt{A}\ \lambda}{V}\right)\right]\right\} . \tag{3.13}$$

IV. VERIFICATION OF THE RANDOM TOUR MODEL

A. DIMENSIONAL ANALYSIS

From equation (3.3) it is clear that ψ must be dimensionless. Now writing ψ as $K\sqrt{A}\,\lambda/V$, we see that K also is a dimensionless coefficient

B. LIMIT OF γ AS $\lambda \to 0$

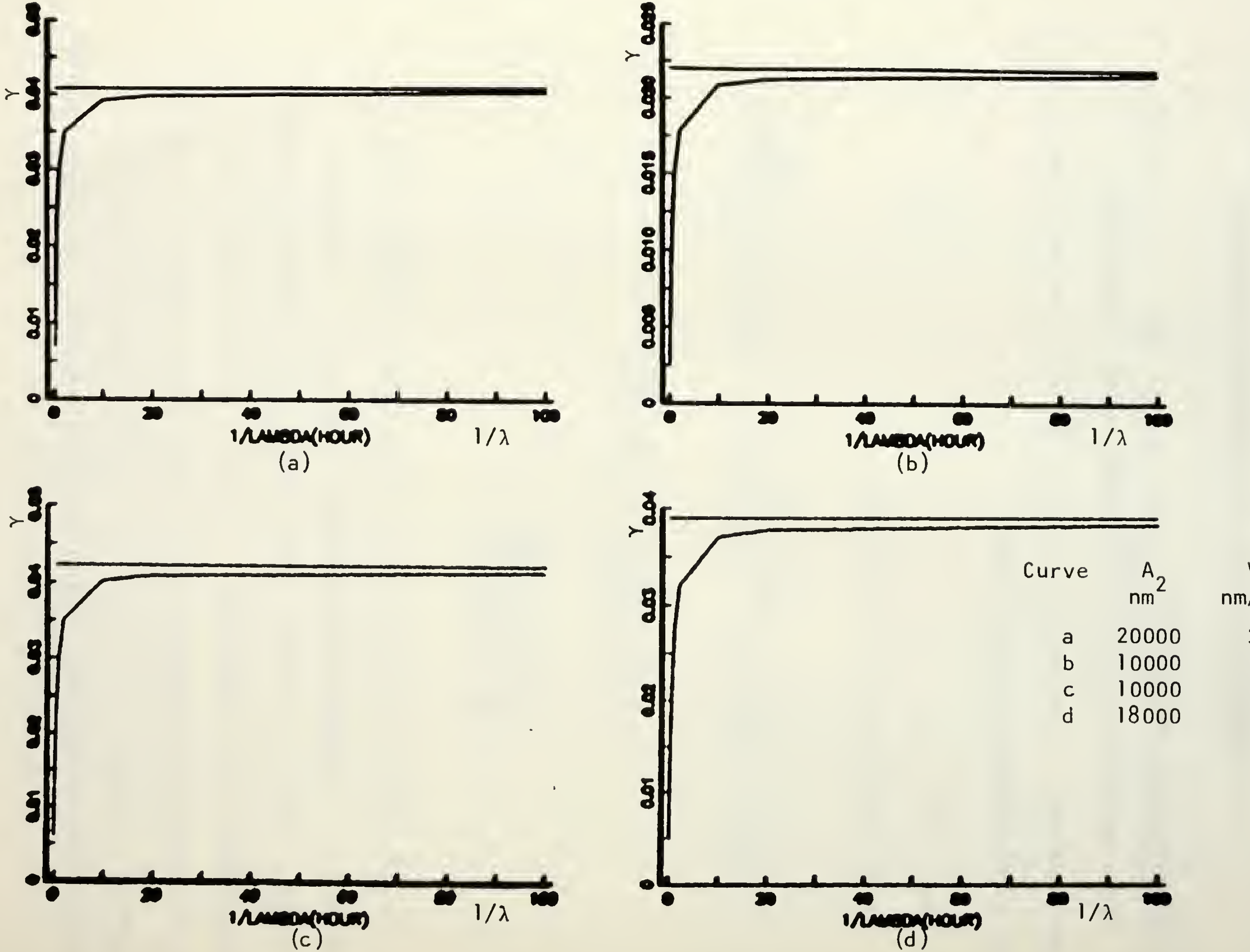
From equation (3.12) we have

$$\lim_{\lambda\to 0} \gamma = \lim_{\lambda\to 0} \left\{ \frac{24.7\ RV^2}{(A - \pi R^2)^{1.5}\lambda} \left[1 - \exp\left(-0.084 \frac{\sqrt{A}\ \lambda}{V}\right)\right]\right\}$$

$$= \frac{24.7\ RV^2}{(A - \pi R^2)^{1.5}} \left(0.084 \frac{\sqrt{A}}{V}\right)$$

$$\simeq \frac{2.0748\ RV\sqrt{A}}{(A - \pi R^2)^{1.5}} \quad . \qquad (4.1)$$

So if equation (3.12) is a reasonable estimate for γ, then as $\lambda \to 0$ RATSIM should give a best fit γ given by equation (4.1). To test this, four groups of simulation experiments were conducted. In each group, values of A, R and V were held constant and λ was varied from 10 to 0.01. Figure 4.1 shows the best fit γ plotted against $1/\lambda$ for each of the simulation groups. Also plotted is a horizontal



Curve	A_2 nm^2	V nm/hr	R nm
a	20000	20	18
b	10000	10	10
c	10000	12	15
d	18000	15	20

Figure 4.1 Variation of γ with $1/\lambda$

line intersecting the Y-axis at the value given by equation (4.1).

For these simulations, it appears that equation (3.12) holds as $\lambda \to 0$.

It is noted that $\lambda = 0$ means that the target never changes course except when reflecting off the area boundaries.

C. LIMIT OF γ AS $\pi R^2 \to A$

From equation (3.12) we see that $\gamma \to \infty$ as $\pi R^2 \to A$, which implies that $PND(t) \to 0$ for $t > 0$. This is as would be expected.

D. ASYMPTOTIC APPROACH TO DIFFUSION MODEL

As stated before, when the ratio of the characteristic length of the search area to the mean segment length of the random tour ($\sqrt{A}/V(1/\lambda)$) becomes large, we find that the random tour model approaches the asymptotically equivalent diffusion model with a diffusion constant V^2/λ. This is consistent with equation (3.12). Since by taking the limits of both sides of (3.12) as ($\sqrt{A}/V(1/\lambda)) \to \infty$ we get

$$\lim_{\sqrt{A}\lambda/V \to \infty} \gamma = \frac{24.7\ RV^2}{(A - \pi R^2)^{1.5}\lambda} .$$

This is the value of β given by equation (2.5) for a diffusion model when the diffusion constant is V^2/λ.

E. ASYMPTOTIC APPROACH TO RANDOM SEARCH MODEL

The random tour model of Koopman [Ref. 7] predicts that the detection rate of a randomly moving target is 2RV/A.

As $\sqrt{A}\,\lambda/V \to 0$, the model presented here results in a detection rate of

$$\frac{2.0748\ RV\sqrt{A}}{(A - \pi R^2)^{3/2}} .$$

For small $\pi R^2/A$ these two expressions are nearly equal.

F. LIMIT OF γ AS $V \to 0$

By taking the limits of both sides of equation (3.12) as $V \to 0$ we get

$$\lim_{V \to 0} \gamma = 0$$

This means that for V = 0,

$$PND(t) = 1 - \pi R^2/A \ , \quad t > 0 \ ,$$

which is as would be expected.

G. SENSITIVITY ANALYSIS

Figure 4.2 illustrates how equation (3.13) behaves as the independent variables A, V, R and λ are varied one at a time. The base case considered was:

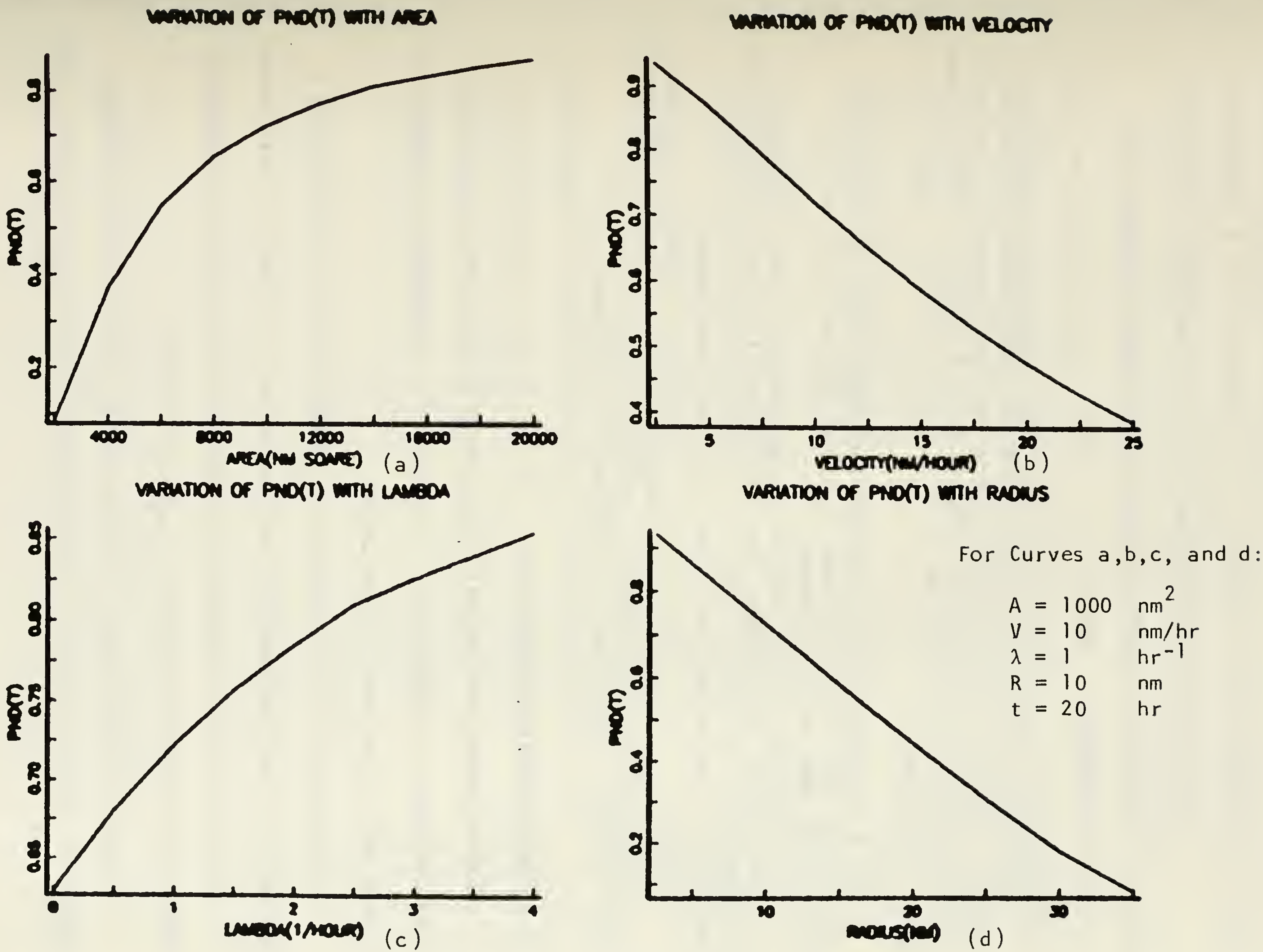


Figure 4.2 Sensitivity Analysis

a) A varies b) V Varies c) λ Varies d) R Varies

$$
\begin{aligned}
A &= 10{,}000 \quad nm^2, \\
V &= 10 \quad nm/hr, \\
\lambda &= 1 \quad hr^{-1}, \\
R &= 10 \quad nm, \\
t &= 20 \quad hr \;.
\end{aligned}
$$

Equation (3.13) is seen to be an increasing function of A and λ, and a decreasing function of V and R. This agrees with intuition.

As A increases, the target has more area in which to hide. So PND will increase.

As V or R increases, the target will be more likely to encounter the detection disk. So PND decreases.

And as λ increases, the target tends to remain closer to its starting position. So PND will increase.

H. FINAL VERIFICATION

There exist no actual data available from real life observations. Therefore, the output of RATSIM was used for final verification of the model.

To achieve this purpose 47 combinations of different values of the independent variables A, V, R and λ were used as input to both simulation program RATSIM and the proposed analytical model given by equation (3.13).

These 47 experiments were classified into four groups, where in each group only one parameter was varied while the others were kept at the base case value ($A = 10000\ nm^2$, $V = 10\ nm/hr$, $R = 10\ nm$ and $\lambda = 1\ hr^{-1}$). The outputs of these different experiments are displayed in Table 4.1.

By looking carefully into the values displayed in Table 4.1, we observe that there is a little difference between the values obtained from simulation and the corresponding values estimated by the proposed analytical model, except for large values of λ ($\lambda > 20$), and for large values of $\pi R^2/A$ ($\pi R^2/A > 0.3$).

So, we can say that the proposed analytical model is reasonable for the realistic values of the problem independent variables (A, V, R and λ) used in antisubmarine warfare (ASW).

Figures 4.3 and 4.4 show a comparison of PND(t) generated by RATSIM and the analytical model for representative values of the independent variables. For many cases the fit is so close that the curves are nearly indistinguishable.

TABLE 4.1

VARIATION OF γ AND $|MAX\ \Delta|^*$ WITH VARIATION IN THE INDEPENDENT VARIABLES

(1) Vary A	(2) Model γ	(3) Simulated γ	(4) $\lvert Max\ \Delta \rvert$
A = 400 nm^2	4.8	2.587	0.08
900	0.388	0.418	0.0305
2000	0.112	0.117	0.0301
4000	0.0455	0.0461	0.0225
6000	0.0276	0.0278	0.0165
8000	0.0194	0.0193	0.011
10000	0.0147	0.0145	0.012
12000	0.0118	0.0117	0.01
14000	0.00972	0.00975	0.015
16000	0.00823	0.00825	0.013
18000	0.0071	0.00721	0.0123
20000	0.00622	0.00625	0.015
25000	0.00468	0.00466	0.016
Vary R			
R = 2.5 nm	0.00352	0.00398	0.047
5	0.0072	0.00738	0.0345
10	0.0147	0.0145	0.012
15	0.0235	0.0229	0.013
20	0.0343	0.0338	0.011
25	0.0487	0.048	0.017
30	0.0693	0.0683	0.0216
35	0.1	0.0986	0.032
40	0.16	0.15	0.041
50	0.706	0.308	0.07

$^*|Max\ \Delta|$: The maximum absolute difference between PND(t) estimated by simulation and PND(t) estimated by the analytical model at the same t, over the whole experiment period (TMAX).

TABLE 4.1 (CONTINUED)

(1)	(2)	(3)	(4)
Vary V			
V = 2.4 nm/hr	0.00156	0.00177	0.0323
5	0.00527	0.00526	0.0244
10	0.0147	0.0145	0.012
15	0.025	0.0251	0.014
20	0.0355	0.0356	0.012
25	0.0462	0.0472	0.013
30	0.057	0.0582	0.021
35	0.06773	0.0689	0.014
40	0.0785	0.0801	0.019
50	0.1	0.11	0.0305
100	0.2088	0.2205	0.0329
200	0.4268	0.446	0.017
Vary λ			
λ = 0.1 hr^{-1}	0.0209	0.0224	0.0217
0.2	0.02004	0.0217	0.0162
0.4	0.0184	0.0196	0.0154
0.6	0.017	0.0168	0.014
0.8	0.0158	0.01579	0.0092
1	0.0147	0.0145	0.012
1.5	0.0137	0.0125	0.016
2	0.0105	0.0102	0.0242
5	0.0051	0.00512	0.037
10	0.00259	0.00294	0.0521
15	0.00173	0.00224	0.054
20	0.0013	0.00687	0.065

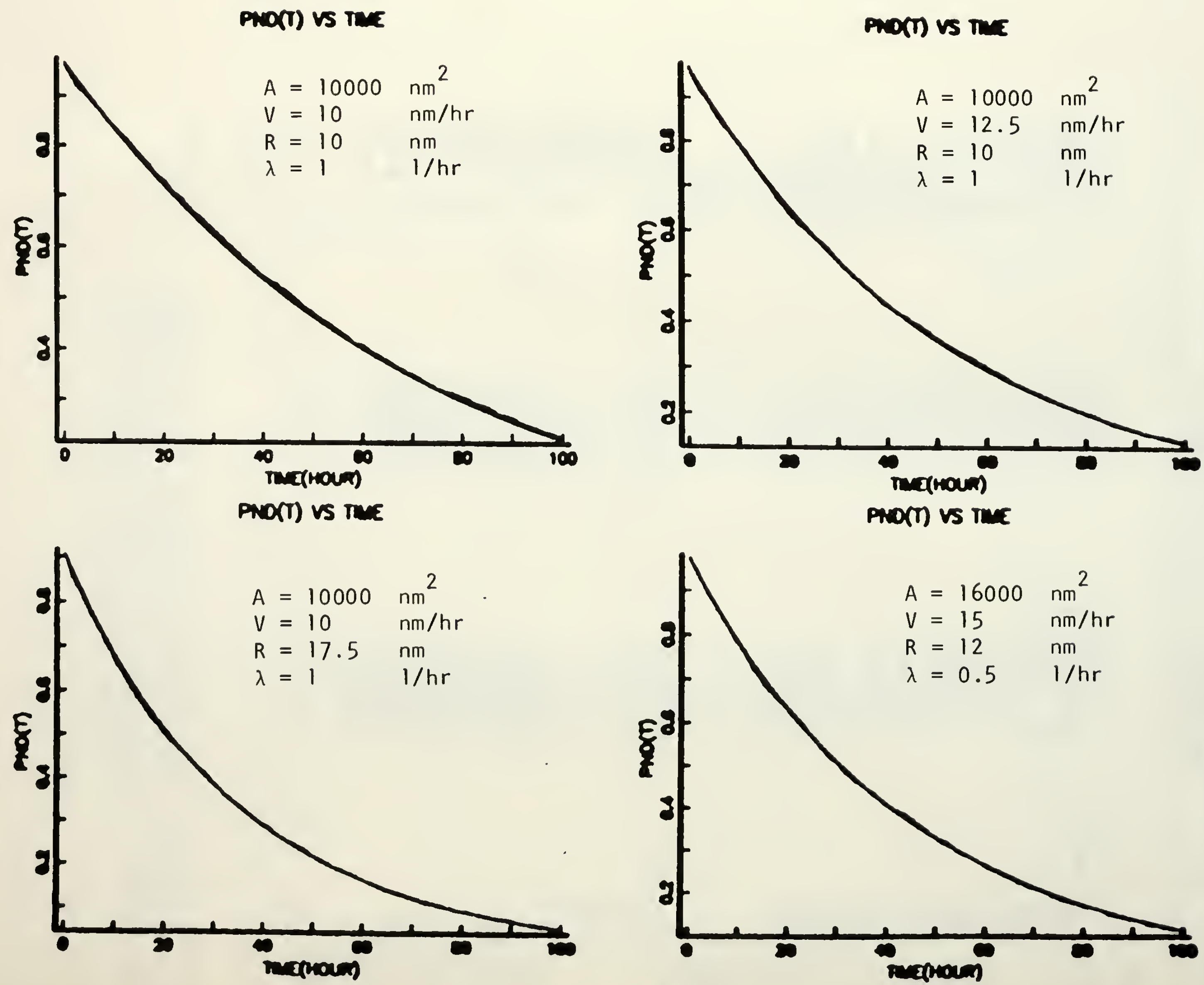


Figure 4.3 Verification Results

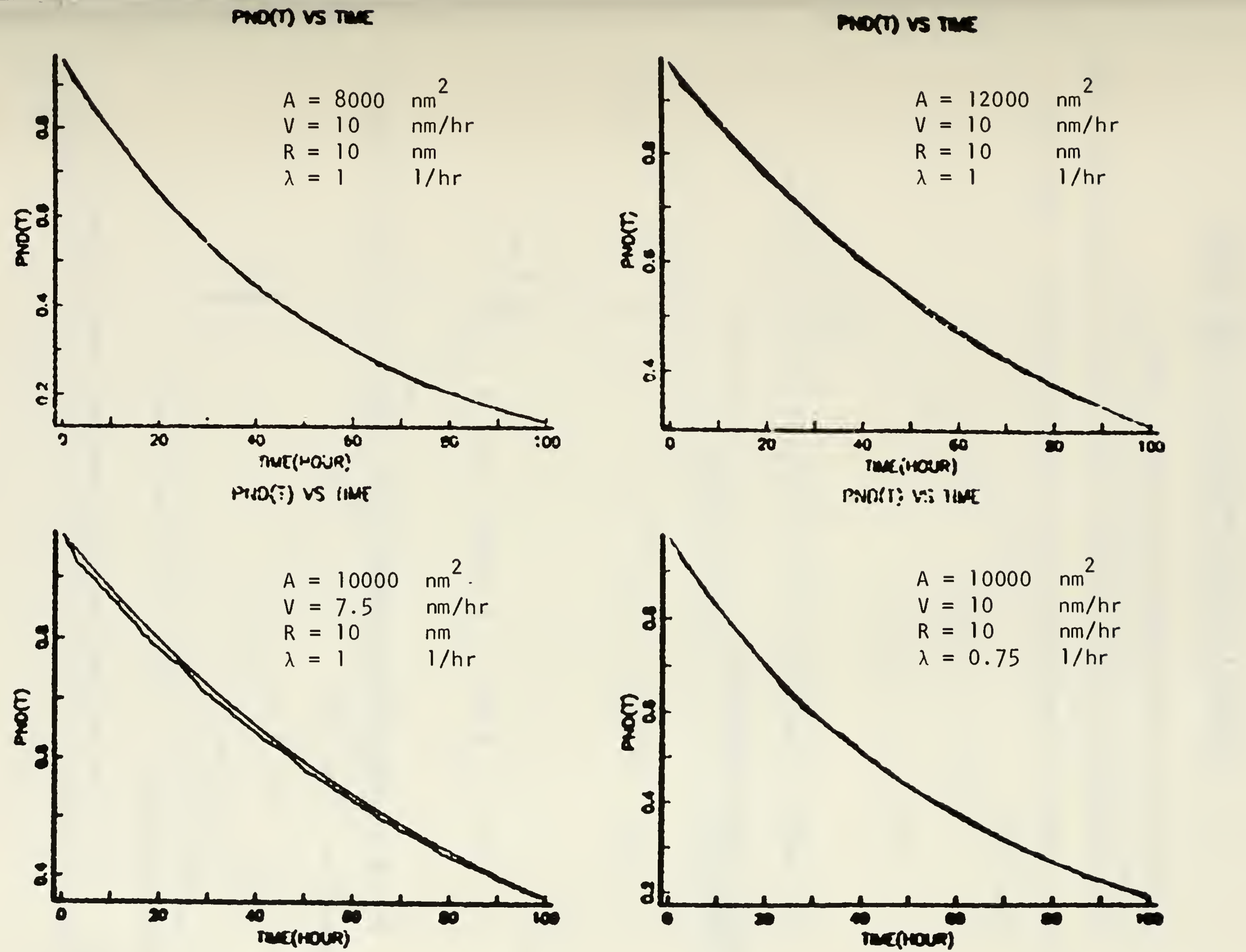


Figure 4.4 Verification Results

V. PROBABILISTIC ANALYSIS OF THE MODEL

A. CUMULATIVE DISTRIBUTION FUNCTION (CDF)

Let T be the random variable for time of detection. And let F(t) be the cumulative distribution function (CDF) for T. That is,

$$P\{T \leq t\} = F(t) \cdot$$

The model presented here implies that F(t) can be closely approximated by

$$F(t) = u(t)\ [1 - \alpha e^{-\gamma t}] \cdot \tag{5.1}$$

where

u(t) is 0 for $t \leq 0$ and 1 for $t > 0$

It is noted that equation (5.1) satisfies the following properties of a CDF:

1) $\lim_{t \to \infty} F(t) = 1$,

2) $F(0) = 0$,

3) $F(t) \geq 0$,

4) F(t) is a non-decreasing function (see Figure 5.1).

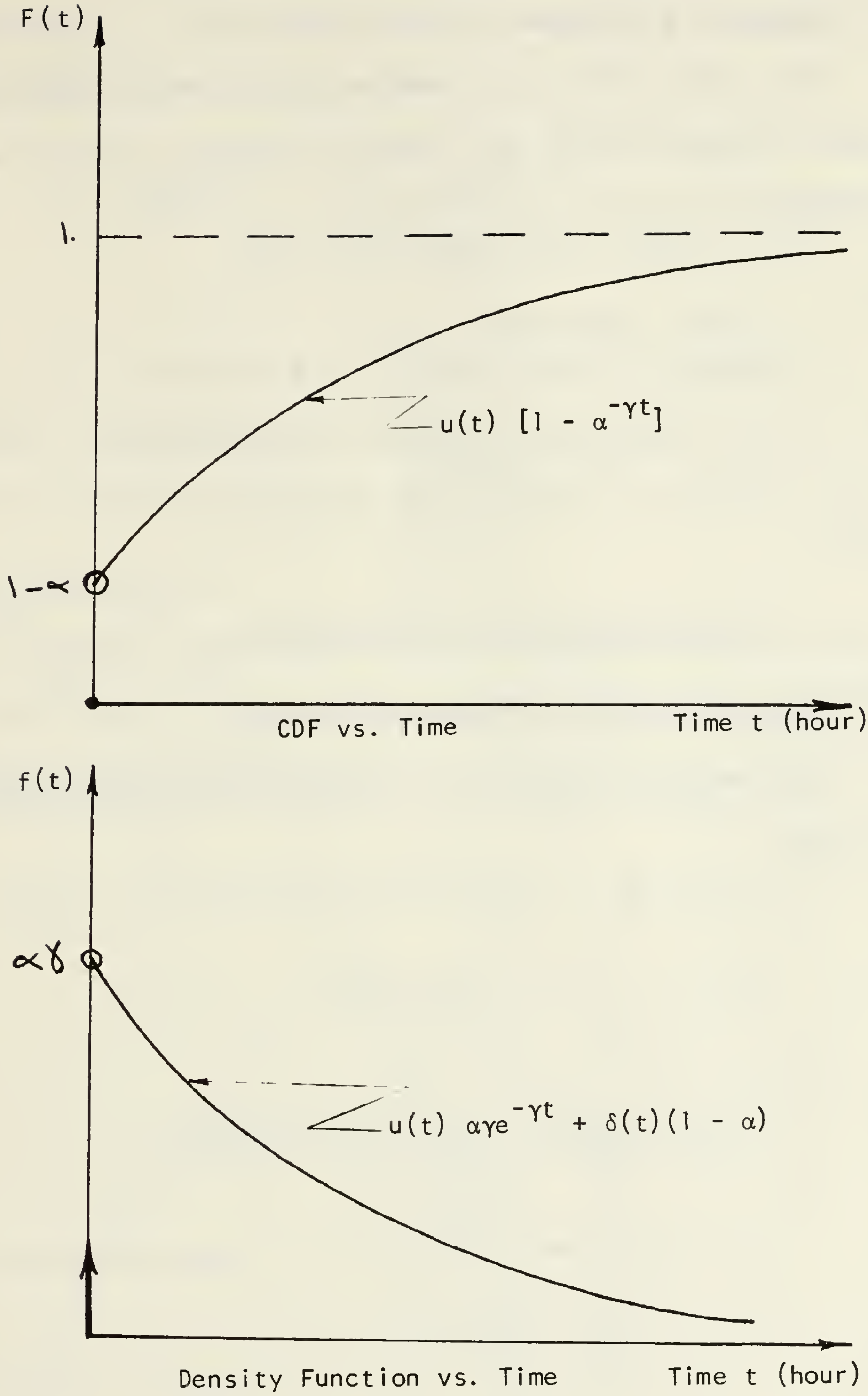


Figure 5.1 Variation of CDF and Density Function with Time

B. DENSITY FUNCTION

By taking the first derivative of F(t) with respect to time, we can derive the density function (f(t)) for T as follows:

$$f(t) = dF(t)/dt$$

$$= u(t)\ (\alpha\gamma e^{-\gamma t}) + \delta(t)(1 - \alpha)\ . \qquad (5.2)$$

where

$\delta(t)$ is the Dirac δ-function.

C. EXPECTED VALUE OF DETECTION TIME

The expected detection time E[T] can be derived as follows:

$$E[T] = \int_0^\infty [1 - F(t)]dt$$

$$= \int_0^\infty [1 - u(t)\ (1 - \alpha e^{-\gamma t})]dt$$

$$= \int_0^\infty \alpha e^{-\gamma t}\ dt$$

$$= \frac{\alpha}{\gamma}\ . \qquad (5.3)$$

Replacing α, γ by their expressions given by equations (3.2), (3.12) respectively we get:

$$E[T] = \left(1 - \frac{\pi R^2}{A}\right)\left[\frac{(A - \pi R^2)^{1.5}\lambda}{24.7\ RV^2[(\exp(-0.084\ \frac{A\ \lambda}{V})) - 1]}\right] \tag{5.4}$$

This equation shows how E[T] varies with the problem independent variables A, R, V and λ. The variation of E[T] with each of these variables is indicated in Figure 5.2.

D. CONDITIONAL CDF

If we assume that there will be no detection at the beginning of the search period, we may derive the following conditional CDF ($F_0(t)$):

$$F_0(t) = P\ \{\text{Detection by time } t \mid \text{no det. at } t = 0^+\}$$

$$= P\ \{T \leq t \mid T > 0\}$$

$$= \frac{P\ (T > 0,\ T < t)}{P\ (T > 0)} = \frac{P(0 < T < t)}{P\ (T > 0)} \tag{5.5}$$

If we substitute $t = 0^+$ in (5.1) we get

$$F(0^+) = 1 - \alpha\ . \tag{5.6}$$

So,

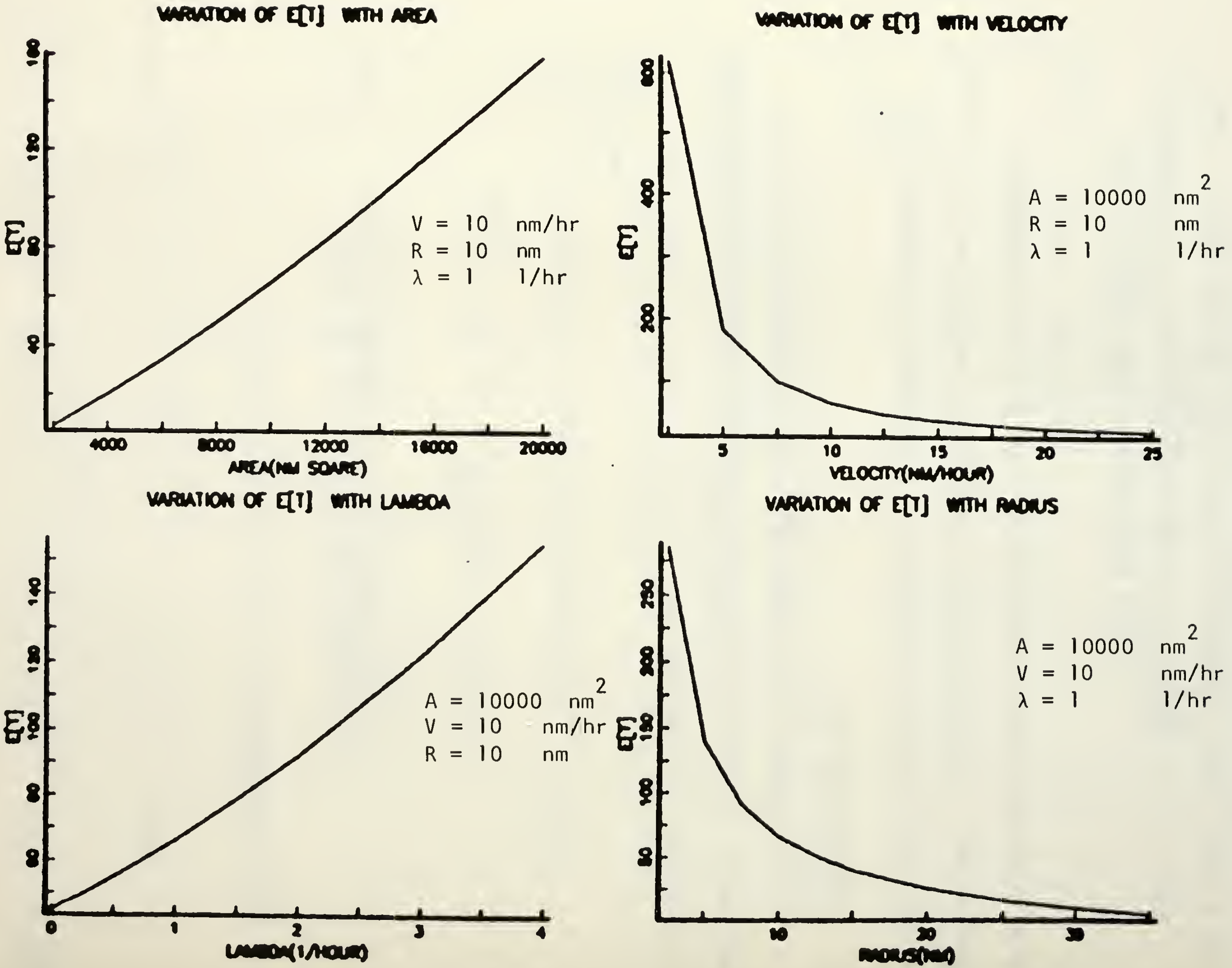


Figure 5.2 Variation of E[T] with A, V, λ and R

$$P\,(0 < T \leq t) = F(t) - F(0^+)$$

$$= (1 - \alpha e^{-\gamma t}) - (1 - \alpha)$$

$$= \alpha(1 - e^{-\gamma t}) \quad . \tag{5.7}$$

Also,

$$P(T > 0) = \overline{F}(0^+)$$

$$= 1 - F(0^+)$$

$$= \alpha \quad . \tag{5.8}$$

Substituting (5.7) and (5.8) into (5.5), we get

$$F_0(t) = (1 - e^{-\gamma t}) \quad . \tag{5.9}$$

This function (5.9) is a CDF for an exponential distribution with parameter γ (an expression for γ is given by (3.12)).

E. CONDITIONAL EXPECTED VALUE OF DETECTION TIME

The conditional expected first detection time $E[T_0]$ can be defined as follows:

$$E[T_0] = E[T \mid \text{no detection at } t = 0]$$

$$= \int_0^{\infty} \overline{F}_0(t)\, dt$$

$$= \frac{1}{\gamma} \quad . \tag{5.10}$$

F. CONDITIONAL DENSITY FUNCTION

Finally, the conditional density function ($f_0(t)$) can be derived as follows:

$$f_0(t) = \frac{dF_0(t)}{dt}$$

$$= \gamma\ e^{-\gamma t} \quad . \tag{5.11}$$

If we compare (5.3) and (5.10), we will observe that

$$\frac{\alpha}{\gamma} < \frac{1}{\gamma} \quad , \text{ since } \alpha < 1 \text{ for } R > 0.$$

This implies that the conditional expected first detection time is greater than the unconditional one. This is reasonable, since in the unconditional case we have an opportunity to detect the target at time 0 . This conclusion is demonstrated clearly by comparing $F(t)$ and $F_0(t)$ as illustrated in Figure 5.3, where we always find that $F_0(t)$ is less than $F(t)$ at any value of t, except at $t = \infty$ where $F(t) = F_0(t) = 1$.

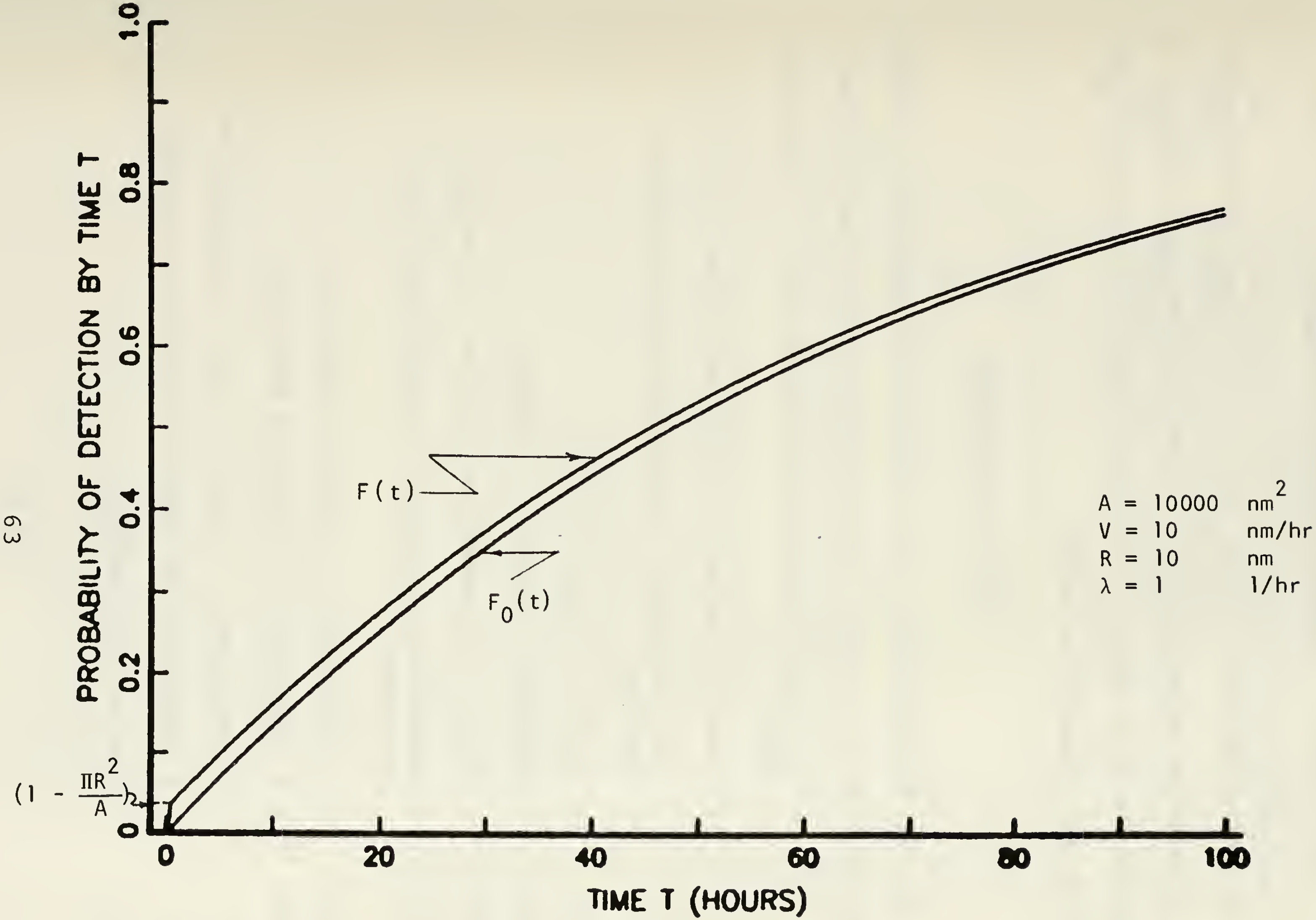


Figure 5.3 Comparison Between CDF (F(t)) and Conditional CDF ($F_0(t)$)

APPENDIX

RATSIM COMPUTER PROGRAM

In order to give access to the logic used in building the simulation model RATSIM, a complete program listing is included in this appendix following the list of variables used in the simulation model.

LIST OF VARIABLES

The variables used in the simulation model are listed below according to their first appearance in the program:

R	=	Radius of searcher detection disk in nautical miles
V	=	Speed of target in nautical miles per hour
Inc	=	Time increment for each discrete step in minutes
REP	=	Number of replications
TMAX	=	Detection period in minutes
SUM(I)	=	Number of detections at time increment I
HIST(I)	=	Accumulative number of detections up to increment I.
POSX	=	X component of target's position
POSY	=	Y component of target's position
XS	=	X component of target's starting position
YS	=	Y component of target's starting position
ANG	=	Course θ in radians

TLEG	=	Time leg for each segment
D	=	Distance between the target location and the center of the detection disk
XN	=	X component of the target's new position
YN	=	Y component of the target's new position
PROBD	=	Probability of detection
PROBS	=	Probability of non-detection
A	=	$1 - \pi R^2/A$ (α)
B	=	Detection rate (γ)

```
C     **********************************************************
C     *PROGRAM NAME:       RATSIM                              *
C     *THIS PROGRAM SIMULATES 2-DIMENSIONAL RANDOM-TOUR MODEL *
C     **********************************************************
       REAL   INC,L,L2,LAMBDA
       INTEGER     REP,CTR,TIME,SUM(8000), HIST(8000)
       DIMENSION    XS(2500),YS(2500),TEXP(3000),TH(3000)
       DIMENSION    TI(8000) ,PROBD(8000),PROBS(8000)
       DIMENSION    Y(8000),Z(8000)
       DOUBLE PRECISION DSEED
       DSEED=89456.D0
       NR=2400
       CALL GGUBS (DSEED,NR, XS)
       DSEED=73452.D0
       NR=2400
       CALL GGUBS (DSEED,NR, YS)
       R=10.
       AREA=10000.
       V=10.
       LAMBDA=1.
       INC=3.
       REP= 2400
       TMAX=100*60
       L=AREA**.5
       L2=L*2
       SER=L/2.
       MAXCTR=INT(TMAX/INC)+1
       DO 10  I=1,MAXCTR
       SUM(I)=0
       HIST(I)=0
10     CONTINUE
        DO 50 I=1,REP
         DSEED=6095.D0*DBLE(FLOAT(I))
         NR=3000
         CALL GGUBS (DSEED,NR,TH)
         DSEED=2211.0D0*DBLE(FLOAT(I))
         XM=1/LAMBDA
         NR=3000
         CALL GGEXN(DSEED,XM,NR,TEXP)
         POSX=XS(I)*L
         POSY=YS(I)*L
         TIME=0
         CTR=1
          DO 40  J=1,3000
          ANG=6.2832 *TH(J)
          TLEG= TEXP(J)*60
          N=INT(TLEG/INC)
           DO 35 M=1,N
             D=(((POSX-SER)**2)+((POSY-SER)**2))**.5
C            CHECK FOR DETECTION
             IF(D.LE.R) GO TO 45
             IF (CTR.GT.MAXCTR)  GO TO 48
C            CURRENT POSITION OF THE TARGET
             XN=POSX+V*(INC/60)*SIN(ANG)
             YN=POSY+V*(INC/60)*COS(ANG)
```

```
C          CHECK FOR REFLECTION
15         IF(XN)  16,16,17
16         XN=-XN
           ANG=6.2832-ANG
17         IF(XN-L)  19,18,18
18         XN=L2-XN
           ANG=6.2832 -ANG
19         IF(YN)    20,20,21
20         YN=-YN
           ANG=3.1416-ANG
21         IF(YN-L)   30,22,22
22         YN=L2-YN
           ANG=3.1416-ANG
30         POSX=XN
           POSY=YN
           CTR=CTR+1
           TIME=TIME+INC
35       CONTINUE
40      CONTINUE
45       INDEX=INT(TIME/INC)+1
         SUM(INDEX)=SUM(INDEX)+1
48     CONTINUE
50    CONTINUE
      HIST(1)=SUM(1)
      PROBD(1)=FLOAT(HIST(1))/FLOAT(REP)
      PROBS(1)=1 -PROBD(1)
      TI(1)=0.0
      DO 300 I=2,MAXCTR
        HIST(I)=SUM(I)+HIST(I-1)
        PROBD(I)=FLOAT(HIST(I))/FLOAT(REP)
        PROBS(I)=1 -PROBD(I)
        TI(I)=FLOAT(I-1)*(INC/60)
        WRITE(6,250) TI(I),PROBS(I)
250     FORMAT(2X,F6.2,2X,F16.11)
300   CONTINUE
C     *****************************************************
C     THE FOLLOWING SUBROUTINE IS TO ESTIMATE THE DETECTION
C     RATE"B" AND THE COEFFICIENT "A" BY USING A NUMERICAL
C     METHOD.
     DO 350 I=1,MAXCTR
         Y(I)=0
         Z(I)=0
350  CONTINUE
     DO 380 I=1,MAXCTR
     IF(PROBS(I).LE.0) GO  TO 390
     Y(I)=PROBS(I)
     Z(I)=TI(I)
380  CONTINUE
390  K=I
      SZ=0
      SZ2=0
      SLY=0
      SLYZ=0
      SLY2=0
        DO 400 J=1,K
```

```
          SZ=SZ+Z(J)
          SZ2=SZ2+Z(J)**2
          IF(Y(J).LE.0) GO TO 500
          SLY=SLY+ALOG(Y(J))
          SLYZ=SLYZ+Z(J)*ALOG(Y(J))
          SLY2=SLY2+ALOG(Y(J))**2
400     CONTINUE
500   U1=(SZ2*SLY)-(SZ*SLYZ)
      G=(K*SZ2)-(SZ**2)
      U2=(K*SLYZ)-(SZ*SLY)
      F=U1/G
      B=U2/G
      A=EXP(F)
      WRITE(6,*)    A,B
      STOP
      END
```

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Thesis
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c.1 A mathematical model for calculating non-detection probability of a random tour target.